Jan 16

Clarification about def joint or mutual independence.

Let $A_1, \ldots, A_n$ are events

i.e. $A_1, \ldots A_n \in \mathcal{E}$.

(I) Events $\{A_1, \ldots A_n\}$ are said to be mutually independ. and if for any subset

$\{i_1, i_2, \ldots, i_k\} \subset \{1, 2, \ldots n\}$,

$$\mathbb{P}(A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k})$$

$$= \mathbb{P}(A_{i_1}) \ldots \mathbb{P}(A_{i_k})$$

(II) Events $\{A_1, \ldots, A_n\}$

are said to be <u>pairwise independent</u> if

$$\mathbb{P}(A_i \cap A_j) = \mathbb{P}(A_i)\,\mathbb{P}(A_j)$$

$$\forall \quad 1 \le i < j \le n.$$

Eg: let there are two kids in a family. Gender of them are equally likely to be boy or girl. and gender of one does not effect the other.

E_1 = one of them is girl

E_2 = both are girls

$$\mathbb{P}(E_2 | E_1) = ?$$

$$\Omega = \{GG, GB, BG, BB\}$$

$$E_1 = \{GG, GB, BG\}$$

$$E_2 = \{GG\}$$

$$\mathbb{P}(E_2 | E_1) = \frac{\mathbb{P}(E_2 \cap E_1)}{\mathbb{P}(E_1)}$$

$$= \frac{1}{3}$$

E_3 = Elder kid is girl

$$\mathbb{P}(E_2 \mid E_3) = ?$$

first kid is elder

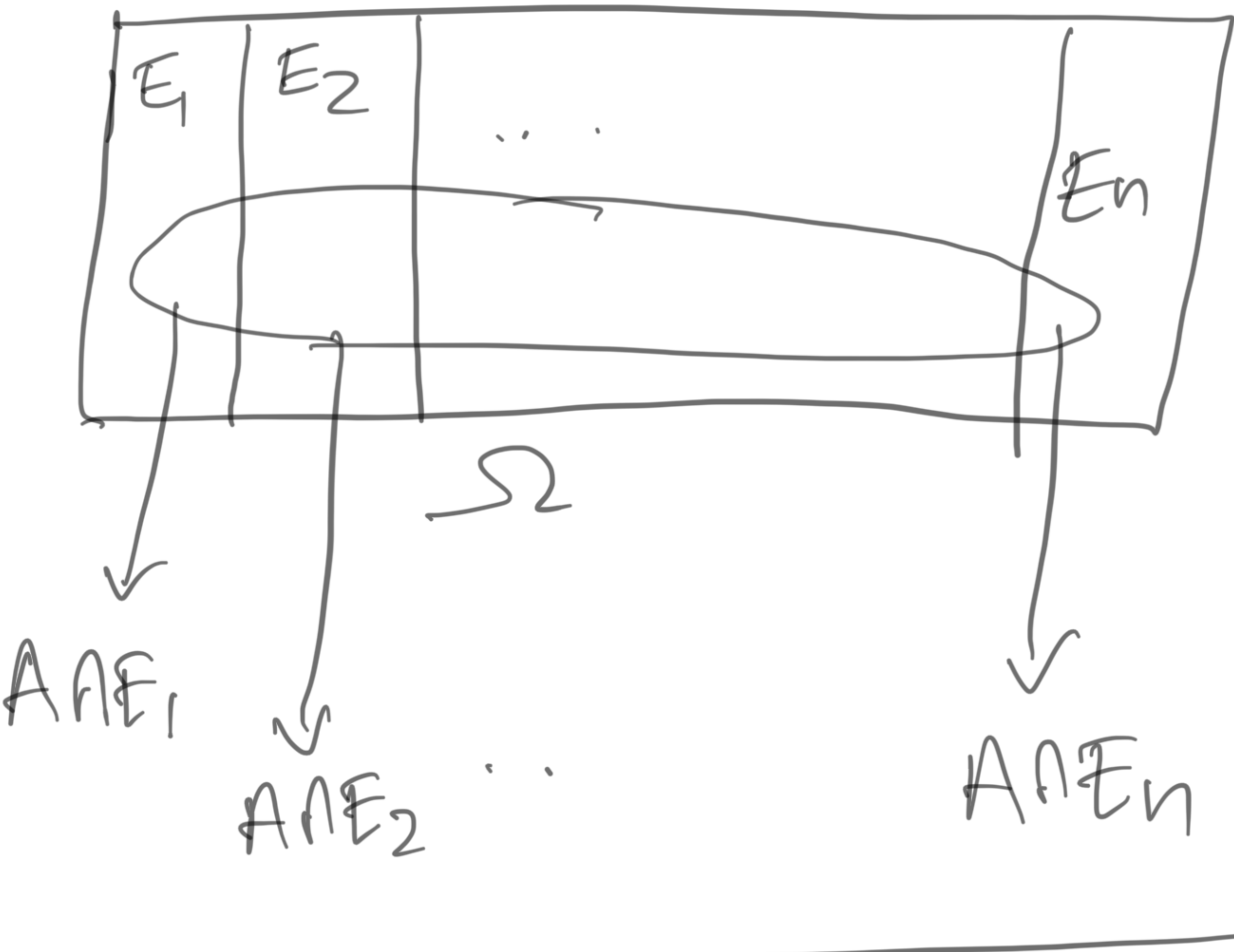
$$E_3 = \{GG, GB\}$$

$$\mathbb{P}(E_2 \mid E_3) = \frac{1}{2}$$

Theorem: Let $E_1, E_2, \ldots, E_n$ is a partion of sample space Ω, and E_i's $\in \mathcal{F}$.

if $A \in \mathcal{F}$, then

$$\mathbb{P}(A) = \sum_{i=1}^{n} \mathbb{P}(A \cap E_i)$$



Lemma: Let $A, B \in \mathcal{F}$.

Then
$$\mathbb{P}(A \cap B) = \underline{\mathbb{P}(A|B) \cdot \mathbb{P}(B)}.$$

Hint: If $\mathbb{P}(B) > 0$
then trivial

If $\mathbb{P}(B) = 0$, then

$$A \cap B \subset B$$

$$\Rightarrow \mathbb{P}(A \cap B) \leq \mathbb{P}(B)$$

$$\Rightarrow \mathbb{P}(A \cap B) = 0$$

Bayes Theorem

Two events

$A, B \in \mathcal{F}$, $\mathbb{P}(A) > 0$, $\mathbb{P}(B) > 0$

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

$$\boxed{\mathbb{P}(A|B) = \frac{\mathbb{P}(B|A) \cdot \mathbb{P}(A)}{\mathbb{P}(B)}}$$

Let $E_1, E_2 \in \mathcal{F}$ such that $E_1 \cap E_2 = \phi$ and $E_1 \cup E_2 = \Omega$.

Further let $A \in \mathcal{F}$.

$$\mathbb{P}(E_1|A) = \frac{\mathbb{P}(A|E_1) \cdot \mathbb{P}(E_1)}{}$$

$$\mathbb{P}(E_1 \mid A) = \frac{\quad}{\mathbb{P}(A)}$$

Note that

$$A = (A \cap E_1) \cup (A \cap E_2)$$

disjoint

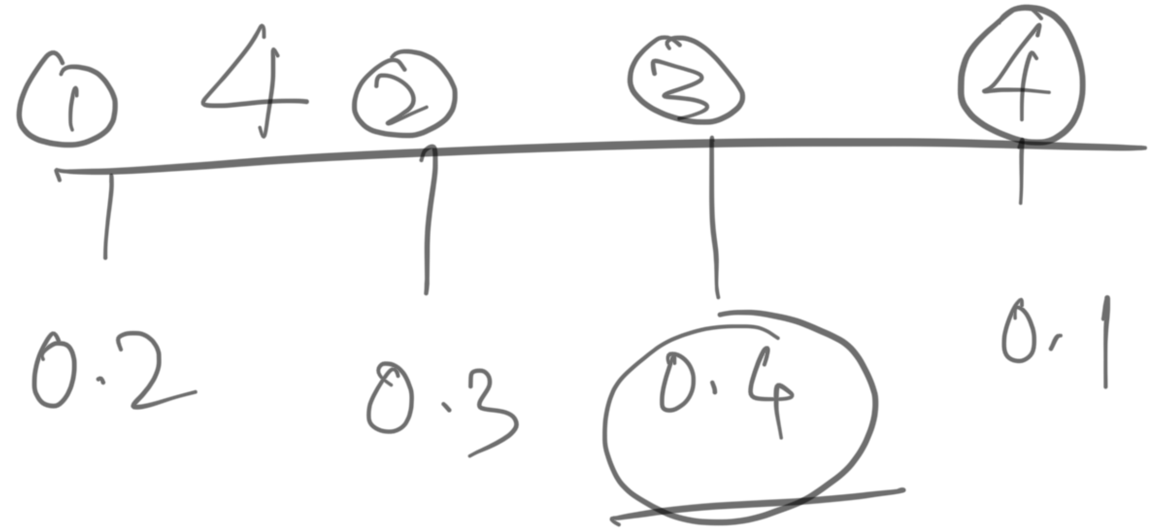
$$\mathbb{P}(A) = \mathbb{P}(A \cap E_1) + \mathbb{P}(A \cap E_2)$$

$$\Rightarrow \mathbb{P}(A) = \mathbb{P}(A \mid E_1) \cdot \mathbb{P}(E_1) + \mathbb{P}(A \mid E_2)\, \mathbb{P}(E_2)$$

So, we get

$\mathbb{P}(E_1 \mid A)$

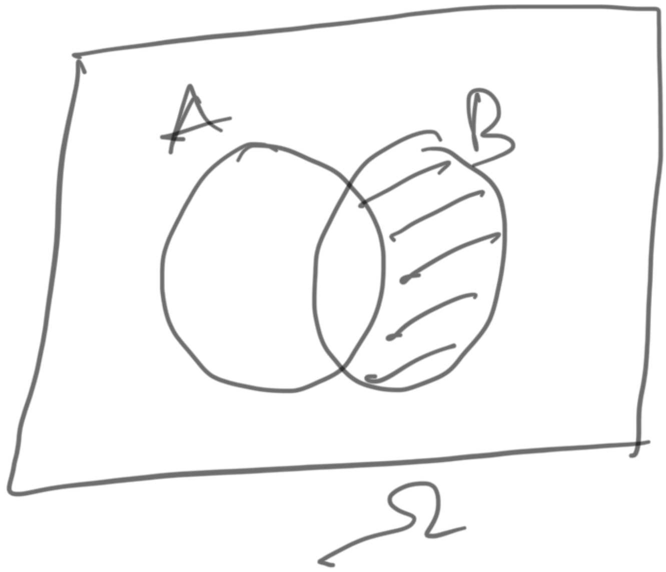
$$= \frac{\mathbb{P}(A|E_1)\cdot\mathbb{P}(E_1)}{\mathbb{P}(A|E_1)\cdot\mathbb{P}(E_1) + \mathbb{P}(A|E_2)\,\mathbb{P}(E_2)}$$



A and B are independent events.

$$\mathbb{P}(A\cap B) = \mathbb{P}(A)\cdot\mathbb{P}(B)$$

Claim: A^c and B are independent



$$
\begin{aligned}
&\mathbb{P}(A^c \cap B) \\
&= \mathbb{P}(B \cap A^c) \\
&= \mathbb{P}(B) - \mathbb{P}(A \cap B) \\
&= \mathbb{P}(B) - \mathbb{P}(A) \cdot \mathbb{P}(B) \\
&= \underbrace{(1 - \mathbb{P}(A))}\ \mathbb{P}(B) \\
&= \mathbb{P}(A^c) \cdot \mathbb{P}(B)
\end{aligned}
$$