

Lecture notes
Calculus

Chapter-2

One variable Differential Calculus

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1 Limits and Continuous functions

Let (a, b) be an interval and $f(x)$ be defined on (a, b) except possibly at $c \in (a, b)$. Then we define the limit of the function f at $x \rightarrow c$ as follows:

Definition 1.0.1. We say that $\lim_{x \rightarrow c} f(x) = L$ if, for every real number $\epsilon > 0$, there exists a real number $\delta > 0$ such that

$$0 < |x - c| < \delta \implies |f(x) - L| < \epsilon. \quad (1.1)$$

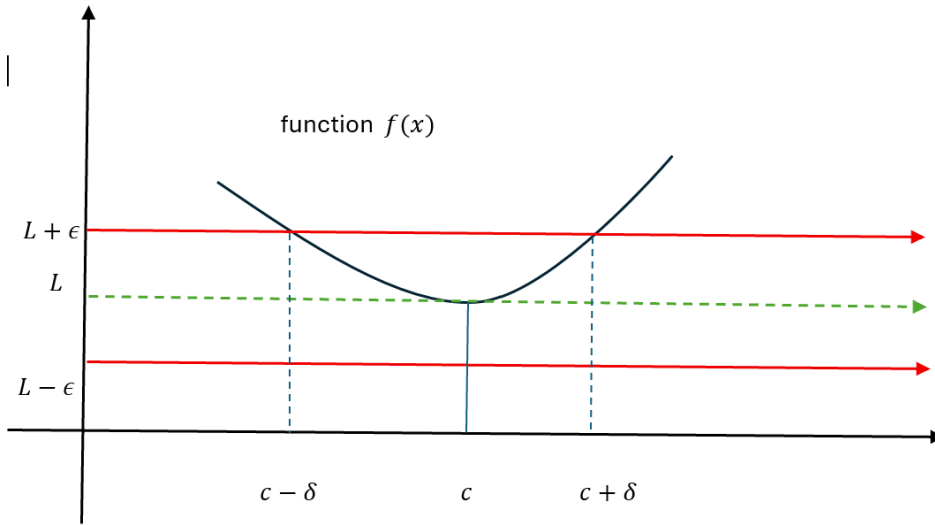


Figure 1: Limit of f at c

Equivalently,

Theorem 1.0.2. The above definition is equivalent to: for any sequence $\{x_n\}$ with $x_n \rightarrow c$, we have $f(x_n) \rightarrow L$ as $n \rightarrow \infty$.

Proof. Suppose $\lim_{x \rightarrow c} f(x)$ exists. Take $\epsilon > 0$ and let $\{x_n\}$ be a sequence converging to c . Then there exists N such that $|x_n - c| < \delta$ for $n \geq N$. Then by the definition $|f(x_n) - L| < \epsilon$. That is, $f(x_n) \rightarrow L$.

For the other side, assume that $x_n \rightarrow c \implies f(x_n) \rightarrow L$. Suppose the limit does not exist. That is, there exists $\epsilon_0 > 0$ such that for any $\delta > 0$, there is $x \in |x - c| < \delta$, for which $|f(x) - L| \geq \epsilon_0$. If we choose $\delta = \frac{1}{n}$ and pick $x_n \in \{x : |x - c| < \frac{1}{n}\}$, then $x_n \rightarrow c$ but $|f(x_n) - L| \geq \epsilon_0$. Not possible.

Theorem 1.0.3. If limit exists, then it is unique.

Proof. Proof is easy.

Example 1.0.4. $\lim_{x \rightarrow 1} \left(\frac{3x}{2} - 1 \right) = \frac{1}{2}$. Let $\epsilon > 0$. We have to find $\delta > 0$ such that (1.1) holds with $L = \frac{1}{2}$. Working backwards,

$$\frac{3}{2}|x - 1| < \epsilon \implies |x - 1| < \frac{2}{3}\epsilon.$$

So if we take $\delta = \frac{2}{3}\epsilon$ we get

$$|x - 1| < \delta \implies |f(x) - L| = \frac{3}{2}|x - 1| < \epsilon.$$

Example 1.0.5. Show that

$$\lim_{x \rightarrow 0} \sin x = 0, \quad \lim_{x \rightarrow 0} \cos x = 1.$$

These limits can be proved using the basic trigonometric relations like

$$\sin \theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

We leave this as an exercise.

Problem: Show that $\lim_{x \rightarrow 0} \sin \left(\frac{1}{x} \right)$ does not exist.



Figure 2: graph of $\sin \left(\frac{1}{x} \right)$

Solution: Consider the sequences $\{x_n\} = \left\{ \frac{1}{n\pi} \right\}$, $\{y_n\} = \left\{ \frac{1}{2n\pi + \frac{\pi}{2}} \right\}$. Then it is easy to see that $x_n, y_n \rightarrow 0$ and $\sin \left(\frac{1}{x_n} \right) \rightarrow 0$, $\sin \left(\frac{1}{y_n} \right) \rightarrow 1$. In fact, for every $c \in [-1, 1]$, we can find a sequence z_n such that $z_n \rightarrow 0$ and $\sin \left(\frac{1}{z_n} \right) \rightarrow c$ as $n \rightarrow \infty$.

1.1 Algebra of limits

Let $c \in (a, b)$ and f, g are real valued functions defined on (a, b) except possibly at c .

Theorem 1.1.1. Suppose $\lim_{x \rightarrow c} f(x) = L$ and $\lim_{x \rightarrow c} g(x) = M$, then

1. $\lim_{x \rightarrow c} (f(x) \pm g(x)) = L \pm M.$
2. $\lim_{x \rightarrow c} (fg)(x) = LM$
3. $\lim_{x \rightarrow c} \left(\frac{f}{g} \right) (x) = \frac{L}{M},$ provided $M \neq 0.$

Proof. We give the proof of (ii). Proof of other assertions are easy to prove. Let $\epsilon > 0$. From the definition of limit, we have $\delta_1, \delta_2, \delta_3 > 0$ such that

$$|x - c| < \delta_1 \implies |f(x) - L| < \frac{1}{2} \implies |f(x)| < N \text{ for some } N > 0,$$

$$|x - c| < \delta_2 \implies |f(x) - L| < \frac{\epsilon}{2M}, \text{ and}$$

$$|x - c| < \delta_3 \implies |g(x) - M| < \frac{\epsilon}{2N}.$$

Hence for $|x - c| < \delta = \min\{\delta_1, \delta_2, \delta_3\}$, we have

$$\begin{aligned} |f(x)g(x) - LM| &\leq |f(x)g(x) - f(x)M| + |f(x)M - LM| \\ &\leq |f(x)||g(x) - M| + M|f(x) - L| < \epsilon. \end{aligned}$$

To prove the second part, W.L.G assume $M > 0$. Then note that there exists an interval $(c - \delta_1, c + \delta_1)$ around c such that $g(x) > \frac{M}{2}$ in $(c - \delta_1, c + \delta_1)$. Then the back calculation

$$\left| \frac{1}{g(x)} - \frac{1}{M} \right| \leq \frac{|g(x) - M|}{M|g(x)|} \leq \frac{2|g(x) - M|}{M^2}$$

in $(c - \delta_1, c + \delta_1)$. Now from the definition of the limit there exists δ_2 such that

$$|x - c| < \delta_2 \implies |g(x) - M| < \frac{M^2 \epsilon}{2}$$

Therefore taking $\delta = \min\{\delta_1, \delta_2\}$, from the above two steps we get

$$|x - c| < \delta \implies \left| \frac{1}{g(x)} - \frac{1}{M} \right| \leq \epsilon.$$

Examples 1.1.2. (i) $\lim_{x \rightarrow 0} x^m = 0$ ($m > 0$). (ii) $\lim_{x \rightarrow 0} x \sin x = 0$. (ii) $\lim_{x \rightarrow 0} |x| \ln(1 + |x|) = 0$.

Next we have the following

Theorem 1.1.3. Sandwich theorem: Suppose that $f(x), g(x)$ and $h(x)$ are such that

1. $f(x) \leq h(x) \leq g(x)$
2. $\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} g(x) = L$

Then $\lim_{x \rightarrow c} h(x) = L$.

Proof. Easy to prove. We leave it as an exercise.

Remark 1.1. Suppose $f(x)$ is bounded in an interval containing c and $\lim_{x \rightarrow c} g(x) = 0$. Then $\lim_{x \rightarrow c} f(x)g(x) = 0$.

This follows from

$$|fg(x)| \leq |f(x)||g(x)| \leq M|g(x) - 0|$$

Examples 1.1.4. (i) $\lim_{x \rightarrow 0} |x| \sin \frac{1}{x} = 0$ (ii) $\lim_{x \rightarrow 0} x[x]$ where $[x]$ is largest integer function.

One sided limits: Let $f(x)$ is defined on (c, b) . The right hand limit of $f(x)$ at c is L , if given $\epsilon > 0$, there exists $\delta > 0$, such that

$$0 < x - c < \delta \implies |f(x) - L| < \epsilon.$$

Notation: $\lim_{x \rightarrow c^+} f(x) = L$. Similarly, one can define the left hand limit of $f(x)$ at b and is denoted by $\lim_{x \rightarrow b^-} f(x) = L$.

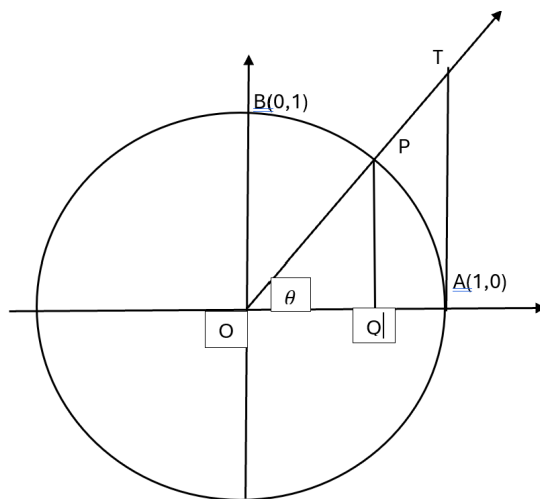
Both theorems above holds for right and left limits. Proof is easy.

At this stage, it is not difficult to prove the following:

Theorem 1.1.5. $\lim_{x \rightarrow c} f(x) = L$ exists $\iff \lim_{x \rightarrow c^+} f(x) = \lim_{x \rightarrow c^-} f(x) = L$.

Problem: Show that $\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$.

Solution: Consider the unit circle centered at $O(0,0)$ and passing through $A(1,0)$ and $B(0,1)$.



Let Q be the projection of P on x -axis and the point T is such that A is the projection of T

on x -axis. Let OT be the ray with $\angle AOT = \theta, 0 < \theta < \pi/2$. Let P be the point of intersection of OT and circle. Then $\triangle OPQ$ and $\triangle OTA$ are similar triangles and hence, Area of $\triangle OAP <$ Area of sector $OAP <$ area of $\triangle OAT$. i.e.,

$$\frac{1}{2} \sin \theta < \frac{1}{2} \theta < \frac{1}{2} \tan \theta$$

dividing by $\sin \theta$, we get $1 > \frac{\sin \theta}{\theta} > \cos \theta$. Now $\lim_{\theta \rightarrow 0^+} \cos \theta = 1$ implies that $\lim_{\theta \rightarrow 0^+} \frac{\sin \theta}{\theta} = 1$.

Now use the fact that $\frac{\sin \theta}{\theta}$ is even function to conclude.

Limits at infinity and infinite limits:

Definition 1.1.6. $f(x)$ has limit L as x approaches $+\infty$, if for any given $\epsilon > 0$, there exists $M > 0$ such that

$$x > M \implies |f(x) - L| < \epsilon.$$

Similarly, one can define limit as x approaches $-\infty$.

Problem: (i) $\lim_{x \rightarrow \infty} \frac{1}{x} = 0$, (ii) $\lim_{x \rightarrow -\infty} \frac{1}{x} = 0$. (iii) $\lim_{x \rightarrow \infty} \sin x$ does not exist.

Solution: (i) and (ii) are easy. For (iii), Choose $x_n = n\pi$ and $y_n = \frac{\pi}{2} + 2n\pi$. Then $x_n, y_n \rightarrow \infty$ and $\sin x_n = 0$, $\sin y_n = 1$. Hence the limit does not exist.

Above two theorems on limits hold in this also.

Definition 1.1.7. (Horizontal Asymptote:) A line $y = b$ is a horizontal asymptote of $y = f(x)$ if either $\lim_{x \rightarrow \infty} f(x) = b$ or $\lim_{x \rightarrow -\infty} f(x) = b$.

Example 1.1.8. (i) $y = 1$ is a horizontal asymptote for $1 + \frac{1}{x+1}$

Definition 1.1.9. (Infinite Limit): A function $f(x)$ approaches ∞ ($f(x) \rightarrow \infty$) as $x \rightarrow x_0$ if, for every real $B > 0$, there exists $\delta > 0$ such that

$$0 < |x - x_0| < \delta \implies f(x) > B.$$

Similarly, one can define for $-\infty$. Also one can define one sided limit of $f(x)$ approaching ∞ or $-\infty$.

Examples 1.1.10. (i) $\lim_{x \rightarrow 0} \frac{1}{x^2}$, (ii) $\lim_{x \rightarrow 0} \frac{1}{x^2} \sin \left(\frac{1}{x} \right)$.

For (i) given $B > 0$, we can choose $\delta \leq \frac{1}{\sqrt{B}}$.

For (ii), choose a sequence $\{x_n\}$ such that $\sin \frac{1}{x_n} = 1$, say $\frac{1}{x_n} = \frac{\pi}{2} + 2n\pi$ and $\frac{1}{y_n} = n\pi$. Then

$\lim_{n \rightarrow \infty} f(x_n) = \frac{1}{x_n^2} \rightarrow \infty$ and $\lim_{n \rightarrow \infty} f(y_n) = 0$, though $x_n, y_n \rightarrow 0$ as $n \rightarrow \infty$.

Definition 1.1.11. (Vertical Asymptote:) A line $x = a$ is a vertical asymptote of $y = f(x)$ if either $\lim_{x \rightarrow a^+} f(x) = \pm\infty$ or $\lim_{x \rightarrow a^-} f(x) = \pm\infty$.

Example 1.1.12. $f(x) = \frac{x+3}{x+2}$.

$x = -2$ is a vertical asymptote and $y = 1$ is a horizontal asymptote.

Next we define the continuous functions and study some basic properties of continuous functions.

1.2 Continuous functions

Definition 1.2.1. A real valued function $f(x)$ is said to be continuous at $x = c$ if

(i) $c \in \text{domain}(f)$

(ii) $\lim_{x \rightarrow c} f(x)$ exists

(iii) The limit in (ii) is equal to $f(c)$.

In short, $\lim_{x \rightarrow c} f(x) = f(c)$.

Sequentially, for every sequence $x_n \rightarrow c$, we must have $f(x_n) \rightarrow f(c)$ as $n \rightarrow \infty$. That is, for a given $\epsilon > 0$, there exists $\delta > 0$ such that

$$|x - c| < \delta \implies |f(x) - f(c)| < \epsilon.$$

Examples 1.2.2.

$$1. f(x) = \begin{cases} x^2 \sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases} \text{ is continuous at } 0.$$

Let $\epsilon > 0$. Then $|f(x) - f(0)| \leq |x^2|$. So it is enough to choose $\delta = \sqrt{\epsilon}$.

$$2. g(x) = \begin{cases} \frac{1}{x} \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases} \text{ is not continuous at } 0.$$

Choose $\frac{1}{x_n} = \frac{\pi}{2} + 2n\pi$. Then $\lim x_n = 0$ and $f(x_n) = \frac{1}{x_n} \rightarrow \infty$.

Theorem 1.2.3. Let $f(x)$ be a continuous function on $[a, b]$ and let $f(c) > 0$ for some $c \in (a, b)$. Then there exists $\delta > 0$ such that $f(x) > 0$ in $(c - \delta, c + \delta)$.

Proof. Let $\epsilon = \frac{1}{2}f(c) > 0$. Since $f(x)$ is continuous at c , there exists $\delta > 0$ such that

$$|x - c| < \delta \implies |f(x) - f(c)| < \frac{1}{2}f(c)$$

i.e., $-\frac{1}{2}f(c) < f(x) - f(c) < \frac{1}{2}f(c)$. Hence $f(x) > \frac{1}{2}f(c)$ for all $x \in (c - \delta, c + \delta)$.

The following theorem on algebra of continuous functions is an easy consequence.

Theorem 1.2.4. Suppose f and g are continuous at c . Then

- (i) $f \pm g$ is also continuous at c
- (ii) fg is continuous at c
- (iii) $\frac{f}{g}$ is continuous at c if $g(c) \neq 0$.

Theorem 1.2.5. Composition of continuous functions is also continuous i.e., if f is continuous at c and g is continuous at $f(c)$ then $g(f(x))$ is continuous at c .

Corollary 1.2.6. If $f(x)$ is continuous at c , then $|f|$ is also continuous at c .

Theorem 1.2.7. If f, g are continuous at c , then $\max(f, g)$ is continuous at c .

Proof. Proof follows from the relation

$$\max(f, g) = \frac{1}{2}(f + g) + \frac{1}{2}|f - g|$$

and the theorems on algebra of limits.

Remark 1.2. For sequences: If f is continuous function and let $x_n \rightarrow x_0$ then $f(x_n) \rightarrow f(x_0)$ (here we assumed that $x_n, x_0 \in \text{dom}(f)$)

Example 1.2.8.

1. $(1 + 1/n)^n \rightarrow e$ implies $(1 + 1/n)^{2n} \rightarrow e^2$ as $f(x) = x^2$ is continuous on $\mathbb{R}$.
2. $n^{1/n} \rightarrow 1$ implies $\frac{n^{1/n}}{1+n^{1/n}} \rightarrow 1/2$ as $f(x) = \frac{x}{1+x}$ is continuous for all $x > 0$.

1.3 Types of discontinuities

There are three types of discontinuities viz removable discontinuity, jump discontinuity and second kind discontinuity or often called essential discontinuity.

Removable discontinuity: $f(x)$ is defined every where in an interval containing a except at $x = a$ and limit exists at $x = a$ OR $f(x)$ is defined also at $x = a$ and limit is NOT equal to function value at $x = a$. Then we say that $f(x)$ has removable discontinuity at $x = a$. These functions can be extended as continuous functions by defining the value of f to be the limit value at $x = a$.

Example 1.3.1. $f(x) = \begin{cases} \frac{\sin x}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$.

Here limit as $x \rightarrow 0$ is 1. But $f(0)$ is defined to be 0.

Jump discontinuity: The left and right limits of $f(x)$ exists but not equal. This type of discontinuities are also called discontinuities of first kind.

Example 1.3.2. $f(x) = \begin{cases} 1 & x \leq 0 \\ -1 & x \geq 0 \end{cases}$.

Easy to see that left and right limits at 0 are different.

Infinite discontinuity: Left or right limit of $f(x)$ is ∞ or $-\infty$. This type discontinuity is also known as essential discontinuity.

Example 1.3.3. $f(x) = \frac{1}{x}$ has infinite discontinuity at $x = 0$.

Discontinuity of second kind: If either $\lim_{x \rightarrow c^-} f(x)$ or $\lim_{x \rightarrow c^+} f(x)$ or both do not exist, then c is called discontinuity of second kind. This type of discontinuity also known as essential discontinuity.

Example 1.3.4. Consider the function

$$f(x) = \begin{cases} 0 & x \in \mathbb{Q} \\ 1 & x \notin \mathbb{Q} \end{cases}$$

Then f does not have left or right limit any point c . Indeed, if $c \in \mathbb{Q}$, then $x_n = c + \frac{1}{n} \in \mathbb{Q}$ and $y_n = c + \frac{\pi}{n} \notin \mathbb{Q}$. For these sequences, we will have, $f(c + \frac{1}{n})$ and $f(c + \frac{\pi}{n})$ converges to different values. If $c \notin \mathbb{Q}$, then choose $x_n \in (c, c + \frac{1}{n}) \cap \mathbb{Q}$ and $y_n = c + \frac{1}{n} \notin \mathbb{Q}$. For these sequences we will again get different limit values.

Properties of continuous functions:

Definition 1.3.5. (Closed set): A subset A of $\mathbb{R}$ is called closed set if A contains all its limit points. (i.e., if $\{x_n\} \subset A$ and $x_n \rightarrow c$, then $c \in A$).

Theorem 1.3.6. Continuous functions on closed, bounded interval is bounded.

Proof. Let $f(x)$ be continuous on $[a, b]$ and let $\{x_n\} \subset [a, b]$ be a sequence such that $|f(x_n)| > n$. Then $\{x_n\}$ is a bounded sequence and hence there exists a subsequence $\{x_{n_k}\}$ which converges to c . Then $f(x_{n_k}) \rightarrow f(c)$, a contradiction to $|f(x_{n_k})| > n_k$.

Theorem 1.3.7. Let $f(x)$ be a continuous function on closed, bounded interval $[a, b]$. Then supremum and infimum of f are achieved in $[a, b]$.

Proof. Above theorem and the completeness of $\mathbb{R}$ implies that Supremum of f is finite. Let $\{x_n\}$ be a sequence such that $f(x_n) \rightarrow \sup f$. Then $\{x_n\}$ is bounded and hence by Bolzano-Weierstrass theorem, there exists a subsequence x_{n_k} such that $x_{n_k} \rightarrow x_0$ for some x_0 . $a \leq x_n \leq b$ implies $x_0 \in [a, b]$. Since f is continuous, $f(x_{n_k}) \rightarrow f(x_0)$. Hence $f(x_0) = \sup f$. The attainment of minimum can be proved by noting that $-f$ is also continuous and $\min f = -\sup(-f)$.

Remark 1.3. Closed and boundedness of the interval is important in the above theorem. Consider the examples (i) $f(x) = \frac{1}{x}$ on $(0, 1)$ (ii) $f(x) = x$ on $\mathbb{R}$.

Theorem 1.3.8. *Let $f(x)$ be a continuous function on $\mathbb{R}$ and let $f(a)f(b) < 0$ for some a, b . Then there exists $c \in (a, b)$ such that $f(c) = 0$.*

Proof. Assume that $f(a) < 0 < f(b)$. Let $S = \{x \in [a, b] : f(x) < 0\}$. Then $[a, a + \delta) \subset S$ for some $\delta > 0$ and S is bounded. Let $c = \sup S$. We claim that $f(c) = 0$. Take $x_n = c + \frac{1}{n}$, then $x_n \notin S$, $x_n \rightarrow c$. Therefore, $f(c) = \lim f(x_n) \geq 0$. On the otherhand, note that $c - \frac{1}{n}$ is NOT supremum. Therefore, there exists a point $y_n \in (c - \frac{1}{n}, c) \cap S$. Then note that $y_n \rightarrow c$, $f(c) = \lim f(y_n) \leq 0$. Hence $f(c) = 0$.

Corollary 1.3.9. Intermediate value theorem: *Let $f(x)$ be a continuous function on $[a, b]$ and let $f(a) < y < f(b)$. Then there exists $c \in (a, b)$ such that $f(c) = y$*

Remark 1.4. *A continuous function assumes all values between its maximum and minimum.*

Problem: (fixed point): Let $f(x)$ be a continuous function from $[0, 1]$ into $[0, 1]$. Then show that there is a point $c \in [0, 1]$ such that $f(c) = c$.

Define the function $g(x) = f(x) - x$. Then $g(0) \geq 0$ and $g(1) \leq 0$. Now Apply Intermediate value theorem.

Application: Root finding: To find the solutions of $f(x) = 0$, one can think of defining a new function g such that $g(x)$ has a fixed point, which in turn satisfies $f(x) = 0$.

Example: (1) $f(x) = x^3 + 4x^2 - 10$ in the interval $[1, 2]$. Define $g(x) = \left(\frac{10}{4+x}\right)^{1/2}$. We can check that g maps $[1, 2]$ into $[1, 2]$. So g has fixed point in $[1, 2]$ which is also solution of $f(x) = 0$. Such fixed points can be obtained as limit of the sequence $\{x_n\}$, where $x_{n+1} = g(x_n)$, $x_0 \in (1, 2)$. Note that

$$|g'(x)| < \frac{1}{2}.$$

By Mean Value Theorem, $\exists z$ (see next section)

$$|x_{n+1} - x_n| = |g'(z)| |x_n - x_{n-1}| \leq \frac{1}{2} |x_n - x_{n-1}|$$

Iterating this, we get

$$|x_{n+1} - x_n| < \frac{1}{2^n} |x_1 - x_0|.$$

Therefore, $\{x_n\}$ is a Cauchy sequence. (see problem after Theorem 1.4.4).

2 Uniformly continuous functions

Definition 2.0.1. A function $f(x)$ is said to be uniformly continuous on a set S , if for given $\epsilon > 0$, there exists $\delta > 0$ such that

$$x, y \in S, |x - y| < \delta \implies |f(x) - f(y)| < \epsilon.$$

Here δ depends only on ϵ , not on x or y .

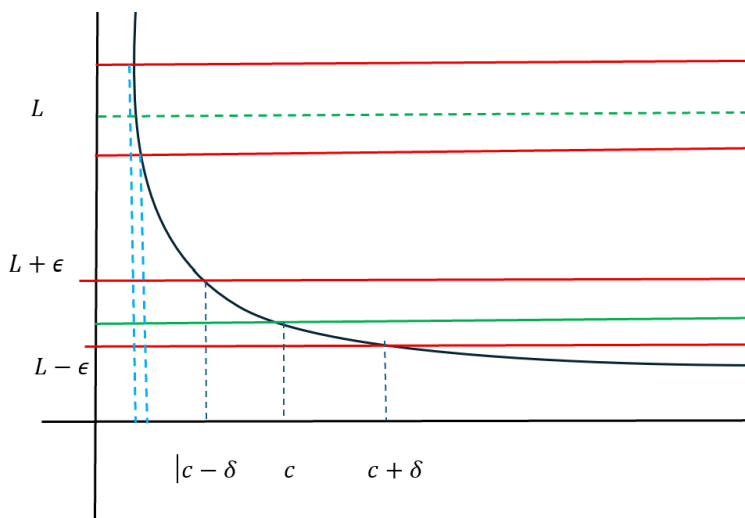


Figure 3: uniform continuity

Examples 2.0.2.

1. $f(x) = x$ is uniformly continuous on $\mathbb{R}$. In this case we can choose $\delta < \epsilon$.

2. $f(x) = x^2$ is uniformly continuous on bounded interval $[-b, b]$.

Note that $|x^2 - y^2| \leq |x + y||x - y| \leq 2b|x - y|$. So one can choose $\delta < \frac{\epsilon}{2b}$.

Remark 2.1. In case of continuous function $f(x) = \frac{1}{x}$ on $(0, 1)$, geometrically one can visualize that the value of δ for the same ϵ depends on the point. Moreover the δ becomes smaller and smaller as we take the point close to 0. But this is only intuition. For getting a mathematical proof, we need the following

Proposition 2.0.3. Proposition: If $f(x)$ is uniformly continuous function $\iff$ for ANY two sequences $\{x_n\}, \{y_n\}$ such that $|x_n - y_n| \rightarrow 0$, we have $|f(x_n) - f(y_n)| \rightarrow 0$ as $n \rightarrow \infty$.

Proof. Suppose not. Then there exists $\{x_n\}, \{y_n\}$ such that $|x_n - y_n| \rightarrow 0$ and $|f(x_n) - f(y_n)| > \eta$ for some $\eta > 0$. Then it is clear that for $\epsilon = \eta$, there is no δ for which $|x - y| < \delta \implies |f(x) - f(y)| < \epsilon$. Because the above sequence satisfies $|x - y| < \delta$, but its image does not.

For converse, assume that for any two sequences $\{x_n\}, \{y_n\}$ such that $|x_n - y_n| \rightarrow 0$ we have $|f(x_n) - f(y_n)| \rightarrow 0$. Suppose f is not uniformly continuous. Then by the definition there exists ϵ_0 such that for any $\delta > 0$, there exist $z, w \in \mathbb{R}$ with $|z - w| < \delta$ for which $|f(z) - f(w)| > \epsilon_0$. Now take $\delta = \frac{1}{n}$ and choose z_n, w_n such that $|z_n - w_n| < \frac{1}{n}$. Then $|f(z_n) - f(w_n)| > \epsilon_0$. A contradiction.

Examples 2.0.4.

1. $f(x) = \frac{1}{x}$ is not uniformly continuous on $(0, 1)$.

Take $x_n = \frac{1}{n+1}, y_n = \frac{1}{n}$, then for n large $|x_n - y_n| \rightarrow 0$ but $|f(x_n) - f(y_n)| = 1$.

2. $f(x) = x^2$ is not uniformly continuous on $\mathbb{R}$.

Take $x_n = n + \frac{1}{n}$ and $y_n = n$. Then $|x_n - y_n| = \frac{1}{n} \rightarrow 0$, but $|f(x_n) - f(y_n)| = 2 + \frac{1}{n^2} > 2$.

Remark 2.2.

1. It is easy to see from the definition that if f, g are uniformly continuous, then $f \pm g$ is also uniformly continuous.
2. If f, g are uniformly continuous, then fg need not be uniformly continuous. This can be seen by noting that $f(x) = x$ is uniformly continuous on $\mathbb{R}$ but x^2 is not uniformly continuous on $\mathbb{R}$.

Theorem 2.0.5. A continuous function $f(x)$ on a closed, bounded interval $[a, b]$ is uniformly continuous.

Proof. Suppose not. Then there exists $\epsilon > 0$ and sequences $\{x_n\}$ and $\{y_n\}$ in $[a, b]$ such that $|x_n - y_n| < \frac{1}{n}$ and $|f(x_n) - f(y_n)| > \epsilon$. But then by Bolzano-Weierstrass theorem, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ that converges to x_0 . Also $y_{n_k} \rightarrow x_0$. Now since f is continuous, we have $f(x_0) = \lim f(x_{n_k}) = \lim f(y_{n_k})$. Hence $|f(x_{n_k}) - f(y_{n_k})| \rightarrow 0$, a contradiction.

Example 2.0.6. $f(x) = \sqrt{x}$ is uniformly continuous on $[0, 1]$

Using mean value theorem of differentiable functions we will show (in lecture 17) that $f(x) = \sqrt{x}, x \in [1, \infty)$ is uniformly continuous. Combining these two intervals we can easily see that $\sqrt{x}$ is uniformly continuous on $[0, \infty)$.

Example 2.0.7. $f(x) = x \sin x$ and $f(x) = \sin x^2$ are uniformly continuous on any closed bounded interval $[a, b]$

Corollary 2.0.8. Suppose $f(x)$ has only removable discontinuities in $[a, b]$. Then $\tilde{f}$, the extension of f , is uniformly continuous.

Example 2.0.9. $f(x) = \frac{\sin x}{x}$ for $x \neq 0$ and 0 for $x = 0$ on $[0, 1]$ has extension

$$\tilde{f}(x) = \begin{cases} \frac{\sin x}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$$

which is uniformly continuous on $[-1, 1]$.

Theorem 2.0.10. *Let f be a uniformly continuous function and let $\{x_n\}$ be a Cauchy sequence. Then $\{f(x_n)\}$ is also a Cauchy sequence.*

Proof. Let $\epsilon > 0$. As f is uniformly continuous, there exists $\delta > 0$ such that

$$|x - y| < \delta \implies |f(x) - f(y)| < \epsilon.$$

Since $\{x_n\}$ is a Cauchy sequence, there exists N such that

$$m, n > N \implies |x_n - x_m| < \delta.$$

Therefore $|f(x_n) - f(x_m)| < \epsilon$ for all $m, n > N$.

Corollary 2.0.11. *Let I be an open interval $f : I \rightarrow \mathbb{R}$ be a uniformly continuous function. Let a be an end point of I . Then limit $\lim_{x \rightarrow a} f(x)$ exists.*

Example 2.0.12. $f(x) = \frac{1}{x^2}$ is not uniformly continuous on $(0, 1)$.

The sequence $x_n = \frac{1}{n}$ is Cauchy but $f(x_n) = n^2$ is not. Hence f cannot be uniformly continuous.

Example 2.0.13. $f(x) = \sin(1/x), x \in (0, 1)$ is not uniformly continuous.

Theorem 2.0.14. *Suppose f is continuous and f' is bounded then f is uniformly continuous.*

Proof. By the mean value theorem and boundedness of f'

$$|f(x) - f(y)| = |f'(c)||x - y| \leq M|x - y|$$

Therefore, given $\epsilon > 0$, we can choose $\delta = \frac{\epsilon}{M}$, to conclude uniform continuity.

Example 2.0.15. $f(x) = \frac{1}{x}, x \in [1, \infty)$ is uniformly continuous.

Problem 1. Show that $f(x) = \sqrt{x}$ is uniformly continuous on $[0, \infty)$.

Solution: We divide the domain into two sub intervals, $I_1 = [0, 1]$ and $I_2 = [1, \infty)$. Then I_1 is closed and bounded interval and f is continuous on I_1 . On I_2 , f' is bounded. Therefore f is uniformly continuous on I_1 and I_2 . For concluding the uniform continuity on the union, we take the $\delta = \min\{\delta_1, \delta_2\}$.

Problem 2. Show that $\cos x$ is uniformly continuous on $\mathbb{R}$

Solution: By the mean value theorem, we get

$$|\cos x - \cos y| \leq |\sin c||x - y| \leq |x - y|$$

therefore for any given ϵ we may choose any $\delta < \epsilon$.

Remark 2.3. We note that if f, g satisfies

$$|f(x) - f(y)| \leq C_1|x - y|; |g(x) - g(y)| \leq C_2|x - y|, \forall x, y$$

Then $h(x) = f(g(x))$ also satisfies

$$|h(x) - h(y)| \leq C_1C_2|x - y|, \forall x, y$$

Examples 2.0.16.

1. $f(x) = |\sin x|$, $x \in \mathbb{R}$ is uniformly continuous.
2. $f(x) = \sqrt{\log x}$, $x > 2$ is uniformly continuous.

3 Differentiability

Definition 3.0.1. A real valued function $f(x)$ is said to be differentiable at x_0 if

$$\lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h} \text{ exists.}$$

This limit is called the derivative of f at x_0 , denoted by $f'(x_0)$.

Examples 3.0.2.

1. $f(x) = x^2$

$$f'(x) = \lim_{h \rightarrow 0} \frac{2xh + h^2}{h} = 2x.$$

2. $f(x) = \sin x$

Using the trigonometric identities and the fact that $\lim_{h \rightarrow 0} \frac{\sin h}{h} = 1$, we get

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{\sin(x + h) - \sin x}{h} \\ &= \lim_{h \rightarrow 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \\ &= (\sin x) \lim_{h \rightarrow 0} \frac{(\cos h - 1)}{h} + (\cos x) \lim_{h \rightarrow 0} \frac{\sin h}{h} \\ &= (\cos x) \lim_{h \rightarrow 0} \frac{-2 \sin^2(h/2)}{h} + \cos x \cdot 1 = 0 + \cos x \end{aligned}$$

Theorem 3.0.3. If $f(x)$ is differentiable at a , then it is continuous at a .

Proof. For $x \neq a$, we may write,

$$f(x) = (x - a) \frac{f(x) - f(a)}{(x - a)} + f(a).$$

Now taking the limit $x \rightarrow a$ and noting that $\lim(x - a) = 0$ and $\lim_{x \rightarrow a} \frac{f(x) - f(a)}{(x - a)} = f'(a)$, we get the result.

Theorem 3.0.4. *Let f, g be differentiable at $c \in (a, b)$. Then $f \pm g, fg$ and $\frac{f}{g}$ (if $g(c) \neq 0$) is also differentiable at c*

Proof. We give the proof for product formula: First note that

$$\frac{(fg)(x) - (fg)(c)}{x - c} = f(x) \frac{g(x) - g(c)}{x - c} + g(c) \frac{f(x) - f(c)}{x - c}.$$

Now taking the limit $x \rightarrow c$, we get the product formula

$$(fg)'(c) = f(c)g'(c) + f'(c)g(c).$$

Since $g(c) \neq 0$ and g is continuous, we get $g(x) \neq 0$ in a small interval around c . Therefore

$$\frac{f}{g}(x) - \frac{f}{g}(c) = \frac{g(c)f(x) - g(c)f(c) + g(c)f(c) - g(x)f(c)}{g(x)g(c)}$$

Hence

$$\frac{(f/g)(x) - (f/g)(c)}{x - c} = \left\{ g(c) \frac{f(x) - f(c)}{x - c} - f(c) \frac{g(x) - g(c)}{x - c} \right\} \frac{1}{g(x)g(c)}$$

Now taking the limit $x \rightarrow c$, we get

$$\left(\frac{f}{g}\right)'(c) = \frac{g(c)f'(c) - f(c)g'(c)}{g^2(c)}.$$

Theorem 3.0.5. *(Chain Rule): Suppose $f(x)$ is differentiable at c and g is differentiable at $f(c)$, then $h(x) := g(f(x))$ is differentiable at c and*

$$h'(c) = g'(f(c))f'(c)$$

Proof. Define the function h as

$$h(y) = \begin{cases} \frac{g(y) - g(f(c))}{y - f(c)} & y \neq f(c) \\ g'(f(c)) & y = f(c) \end{cases}$$

Then the function h is continuous at $y = f(c)$ and so $h(f(x)) \rightarrow h(f(c)) = g(f(c))$. Also

$$g(y) - g(f(c)) = h(y)(y - f(c))$$

Therefore,

$$\frac{h(x) - h(c)}{x - c} = \frac{g(f(x)) - g(f(c))}{x - c} = h(f(x)) \frac{f(x) - f(c)}{x - c}.$$

Now taking limit $x \rightarrow c$, we get the required formula.

Problem 3. Find the derivative of

$$(i) \sin(x^2) \quad (ii) e^{\sin x} \quad (iii) \log(e^x) \quad (iv) \sqrt{\log x}$$

Local extremum: A point $x = c$ is called local maximum of $f(x)$, if there exists $\delta > 0$ such that

$$0 < |x - c| < \delta \implies f(c) \geq f(x).$$

Similarly, one can define local minimum: $x = b$ is a local minimum of $f(x)$ if there exists $\delta > 0$ such that

$$0 < |x - b| < \delta \implies f(b) \leq f(x).$$

Theorem 3.0.6. Let $f(x)$ be a differentiable function on (a, b) and let $c \in (a, b)$ is a local maximum of f . Then $f'(c) = 0$.

Proof. Let δ be as in the above definition. Then

$$x \in (c, c + \delta) \implies \frac{f(x) - f(c)}{x - c} \leq 0$$

$$x \in (c - \delta, c) \implies \frac{f(x) - f(c)}{x - c} \geq 0.$$

Now taking the limit $x \rightarrow c$, we get $f'(c) = 0$.

Theorem 3.0.7. Rolle's Theorem: Let $f(x)$ be a continuous function on $[a, b]$ and differentiable on (a, b) such that $f(a) = f(b)$. Then there exists $c \in (a, b)$ such that $f'(c) = 0$.

Proof. If $f(x)$ is constant, then it is trivial. Suppose $f(x_0) > f(a)$ for some $x_0 \in (a, b)$, then f attains maximum at some $c \in (a, b)$. Other possibilities can be worked out similarly.

Geometrically, for a non-constant differentiable function if $f(a) = f(b)$ at some a, b then there exists a point $c \in (a, b)$ such that the slope at c is parallel to x -axis.

Theorem 3.0.8. Mean-Value Theorem (MVT): Let f be a continuous function on $[a, b]$ and differentiable on (a, b) . Then there exists $c \in (a, b)$ such that

$$f(b) - f(a) = f'(c)(b - a).$$

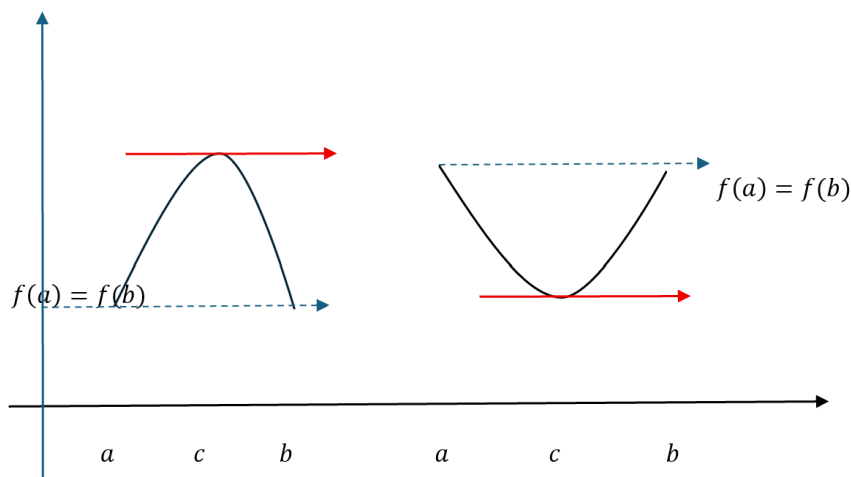


Figure 4: Rolle's theorem

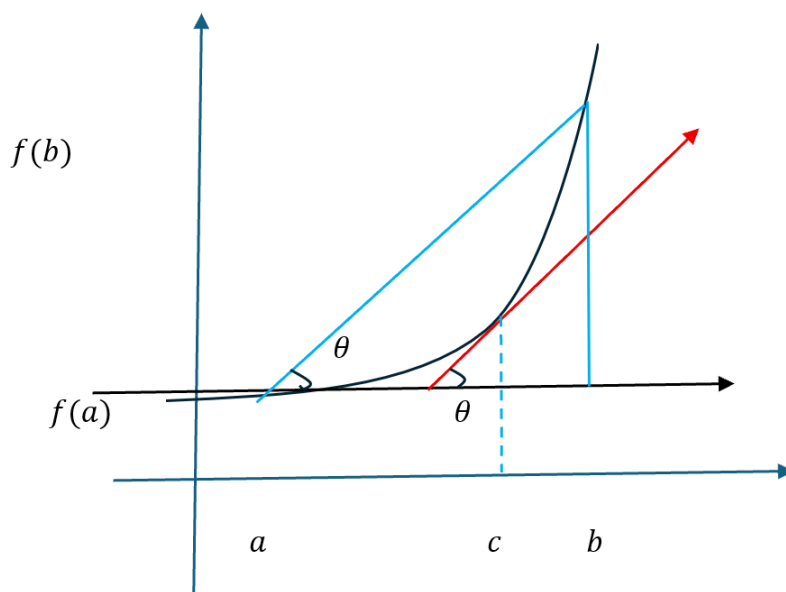


Figure 5: Mean value theorem

Proof. Geometrically, from the figure 5,

$$\frac{f(b) - f(a)}{b - a} = \tan \theta$$

But $\tan \theta$ is equal to $f'(c)$ for some $c \in (a, b)$.

Formally, we can prove this by taking $l(x)$ the straight line joining $(a, f(a))$ and $(b, f(b))$. Con-

sider the function $g(x) = f(x) - l(x)$. Then $g(a) = g(b) = 0$. Hence by Rolle's theorem

$$0 = g'(c) = f'(c) - \frac{f(b) - f(a)}{b - a}$$

Corollary 3.0.9. *If f is a differentiable function on (a, b) and $f' = 0$, then f is constant.*

Proof. By mean value theorem $f(x) - f(y) = 0$ for all $x, y \in (a, b)$.

Problem 4. *Suppose a sequence defined by $x_{n+1} = f(x_n), n \geq 1, x_0 \in \text{domain of } f$ and if $|f'(x)| \leq \alpha < 1$ for all x , then show that $\{x_n\}$ converges.*

As a consequence of the MVT,

$$|x_{n+1} - x_n| \leq |f(x_n) - f(x_{n-1})| \leq |f'(c)| |x_n - x_{n-1}| < \alpha |x_n - x_{n-1}|$$

Therefore $\{x_n\}$ is a Cauchy sequence and hence converges.

Example 3.0.10. *Let $g(x) = \left(\frac{10}{4+x}\right)^{1/2}$ on $[1, 2]$ and the sequence defined by $x_1 \in (1, 2)$, $x_{n+1} = g(x_n), n \in \mathbb{N}$ converges.*

Definition 3.0.11. *A function $f(x)$ is strictly increasing on an interval I , if for $x, y \in I$ with $x < y$ we have $f(x) < f(y)$. We say f is strictly decreasing if $x < y$ in I implies $f(x) > f(y)$.*

Theorem 3.0.12. *A differentiable function f is (i) strictly increasing in (a, b) if $f'(x) > 0$ for all $x \in (a, b)$. (ii) strictly decreasing in (a, b) if $f'(x) < 0$.*

Proof. Choose x, y in (a, b) such $x < y$. Then by MVT, for some $c \in (x, y)$

$$\frac{f(x) - f(y)}{x - y} = f'(c) > 0.$$

Hence $f(x) < f(y)$.

Example 3.0.13. *Find the intervals where the following function is increasing and decreasing*

$$f(x) = x^2 \left(x - \frac{3}{2}\right)$$

4 Taylor's theorem

Let f be a k times differentiable function on an interval I of $\mathbb{R}$. We want to approximate this function by a polynomial $P_n(x)$ such that $P_n(a) = f(a)$ at a point a . Moreover, if the derivatives of f and P_n also equal at a then we see that this approximation becomes more accurate in a neighbourhood of a . So the best coefficients of the polynomial can be calculated using the relation $f^{(k)}(a) = P_n^{(k)}(a), k = 0, 1, 2, \dots, n$. The best is in the sense that if $f(x)$ itself is a

polynomial of degree less than or equal to n , then both f and P_n are equal. This implies that the polynomial is $\sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x-a)^k$. Then we write $f(x) = P_n(x) + R_n(x)$ in a neighbourhood of a . From this, we also expect the $R_n(x) \rightarrow 0$ as $x \rightarrow a$. In fact, we have the following theorem known as **Taylor's theorem**:

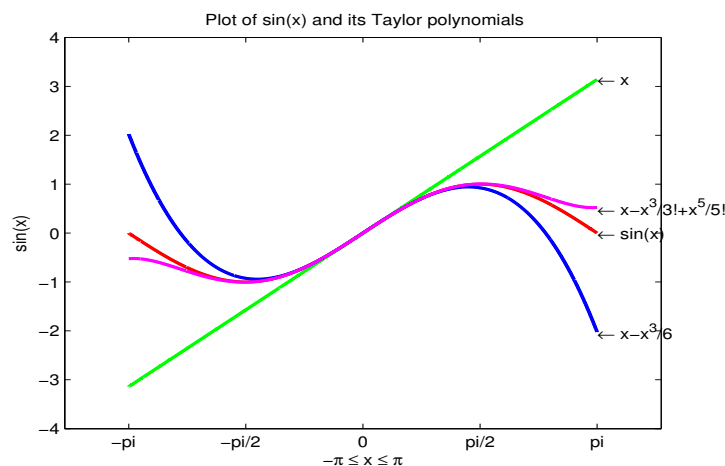


Figure 6: Approximation of $\sin x$ by Taylor's polynomials

Theorem 4.0.1. Let $f(x)$ and its derivatives of order m are continuous and $f^{(m+1)}(x)$ exists in a neighbourhood of $x = a$. Then there exists $c \in (a, x)$ (or $c \in (x, a)$) such that

$$f(x) = f(a) + f'(a)(x-a) + \dots + f^{(m)}(a) \frac{(x-a)^m}{m!} + R_m(x)$$

where $R_m(x) = \frac{f^{(m+1)}(c)}{(m+1)!} (x-a)^{m+1}$.

Proof. Define the functions F and g as

$$F(y) = f(x) - f(y) - f'(y)(x-y) - \dots - \frac{f^{(m)}(y)}{m!} (x-y)^m,$$

$$g(y) = F(y) - \left(\frac{x-y}{x-a} \right)^{m+1} F(a).$$

Then it is easy to check that $g(a) = 0$. Also $g(x) = F(x) = f(x) - f(x) = 0$. Therefore, by Rolle's theorem, there exists some $c \in (a, x)$ such that

$$g'(c) = 0 = F'(c) + \frac{(m+1)(x-c)^m}{(x-a)^{m+1}} F(a).$$

On the other hand, from the definition of F ,

$$F'(c) = -\frac{f^{(m+1)}(c)}{m!}(x-c)^m.$$

Hence $F(a) = \frac{(x-a)^{m+1}}{(m+1)!}f^{(m+1)}(c)$ and the result follows.

Examples 4.0.2.

1. If $f(x) = e^x$ and $a = 0$

Then $f^{(n)}(0) = 1$ for all n . Hence the Taylor polynomial is $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!}$. The remainder term is $R_n(x) = e^c \frac{x^{n+1}}{(n+1)!}$, for some $c \in (0, x)$ or $(x, 0)$ depending on the sign of x .

2. If $f(x) = \sin x$ and $a = 0$

$$\text{Then } f^{(n)}(0) = \begin{cases} 0 & n = 0, 2, 3, 4, \dots \\ 1 & n = 1, 5, 7, \dots \\ -1 & n = 3, 7, \dots \end{cases}$$

Hence the Taylor's polynomial is $x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots + (-1)^n \frac{x^{(2n+1)}}{(2n+1)!}$. The remainder term is $R_{2n+1}(x) = \frac{x^{2n+2}}{(2n+2)!} \sin(c + \frac{(2n+2)\pi}{2})$, $c \in (0, x)$ or $(x, 0)$ depending on the sign of x .

3. If $f(x) = \cos x$ and $a = 0$.

Then the Taylor polynomial is $1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots + \frac{(-1)^n x^{2n}}{(2n)!}$. The remainder term is $R_{2n}(x) = \cos(c + \frac{2n+1\pi}{(2n+1)!})$, $c \in (0, x)$ or $(x, 0)$ depending on the sign of x .

Problem 5. Find the optimal (least) order n of Taylor polynomial P_n , about $x = 0$ to approximate e^x in $(-1, 1)$ so that the maximum absolute error is not more than 0.005

Solution: We know that $p_n(x) = 1 + x + \dots + \frac{x^n}{n!}$. The maximum error in $[-1, 1]$ is

$$|R_n(x)| \leq \frac{1}{(n+1)!} \max_{[-1, 1]} |x|^{n+1} e^x \leq \frac{e}{(n+1)!}.$$

So n is such that $\frac{e}{(n+1)!} \leq 0.005$ or $n \geq 5$. so the required optimal order is of polynomial 5.

Problem 6. Find the interval of validity (rounded off to 2 decimal places) when we approximate $\cos x$ with 2nd order polynomial with maximum absolute error less than 10^{-4} .

Solution: Taylor polynomial of degree 2 for $\cos x$ is $1 - \frac{x^2}{2}$. So the remainder is $(\sin c) \frac{x^3}{3!}$. Since $|\sin c| \leq 1$, the error will be atmost 10^{-4} if $\frac{x^3}{3!} \leq 10^{-4}$. Solving this gives $|x| < 0.084$.

Maxima, Minima and second Derivative test

Definition 4.0.3. A point $x = a$ is called critical point of the function $f(x)$ if $f'(a) = 0$.

Second derivative test: A point $x = a$ is a local maxima if $f'(a) = 0, f''(a) < 0$.

Suppose $f(x)$ is continuously differentiable in an interval around $x = a$ and let $x = a$ be a critical point of f . Then $f'(a) = 0$. By Taylor's theorem around $x = a$, there exists, $c \in (a, x)$ (or $c \in (x, a)$),

$$f(x) - f(a) = \frac{f''(c)}{2}(x - a)^2.$$

If $f''(a) < 0$. Then by the continuity of f'' , there exists $\delta > 0$ such that $f''(c) < 0$ in $|x - a| < \delta$. Hence $f(x) < f(a)$ in $|x - a| < \delta$, which implies that $x = a$ is a local maximum.

Similarly, one can show the following for local minima: $x = a$ is a local minima if $f'(a) = 0, f''(a) > 0$.

Also the above observations show that if $f'(a) = 0, f''(a) = 0$ and $f^{(3)}(a) \neq 0$, then the sign of $f(x) - f(a)$ depends on $(x - a)^3$. i.e., it has no constant sign in any interval containing a . Such point is called point of inflection or saddle point.

We can also derive that if $f'(a) = f''(a) = f^{(3)}(a) = 0$, then we again have $x = a$ is a local minima if $f^{(4)}(a) > 0$ and is a local maxima if $f^{(4)}(a) < 0$.

Summarizing the above, we have:

Theorem 4.0.4. *Let f be a real valued function that is differentiable $2n$ times and $f^{(2n)}$ is continuous at $x = a$. Then*

1. *If $f^{(k)}(a) = 0$ for $k = 1, 2, \dots, 2n - 1$ and $f^{(2n)}(a) > 0$ then a is a point of local minimum of $f(x)$*
2. *If $f^{(k)}(a) = 0$ for $k = 1, 2, \dots, 2n - 1$ and $f^{(2n)}(a) < 0$ then a is a point local maximum of $f(x)$.*
3. *If $f^{(k)}(a) = 0$ for $k = 1, 2, \dots, 2n - 2$ and $f^{(2n-1)}(a) \neq 0$, then a is point of inflection. i.e., f has neither local maxima nor local minima at $x = a$.*

L'Hospitals Rule:

Suppose $f(x)$ and $g(x)$ are differentiable n times, $f^{(n)}, g^{(n)}$ are continuous at a and $f^{(k)}(a) = g^{(k)}(a) = 0$ for $k = 0, 1, 2, \dots, n - 1$. Also if $g^{(n)}(a) \neq 0$. Then by Taylor's theorem,

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f^{(n)}(c)}{g^{(n)}(c)} = \frac{f^{(n)}(a)}{g^{(n)}(a)}$$

In the above, we used the fact that $g^{(n)}(x) \neq 0$ "near $x = a$ " and $g^{(n)}(c) \rightarrow g^{(n)}(a)$ as $x \rightarrow a$.

Similarly, we can derive a formula for limits as x approaches infinity by taking $x = \frac{1}{y}$.

$$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \lim_{y \rightarrow 0} \frac{f(1/y)}{g(1/y)} = \lim_{y \rightarrow 0} \frac{(-1/y^2)f'(1/y)}{(-1/y^2)g'(1/y)} = \lim_{x \rightarrow \infty} \frac{f'(x)}{g'(x)}$$

Taylor Series:

Suppose f is infinitely differentiable at a and if the remainder term in the Taylor's formula, $R_n(x) \rightarrow 0$ as $n \rightarrow \infty$. Then we say that the Taylor Series converges to the function $f(x)$ at the point x . So we may write

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n. \quad (4.2)$$

For a fixed x , the above infinite sum is a series of real numbers. One can independently check the convergence of such series by the convergence tests. However that does not confirm if the series is equal to $f(x)$. The series on the RHS of (4.2) is called Taylor series of $f(x)$ about the point a .

In some cases it is easy to verify this. For example,

Suppose there exists $C = C(x) > 0$, independent of n , such that $|f^{(n)}(x)| \leq C(x)$. Then $|R_n(x)| \rightarrow 0$ if $\lim_{n \rightarrow \infty} \frac{|x-a|^{n+1}}{(n+1)!} = 0$. For any fixed x and a , we can always find N such that $|x-a| < N$. Let $q := \frac{|x-a|}{N} < 1$. Then

$$\begin{aligned} \left| \frac{(x-a)^{n+1}}{(n+1)!} \right| &= \left| \frac{x-a}{1} \right| \left| \frac{x-a}{2} \right| \cdots \left| \frac{x-a}{N-1} \right| \left| \frac{x-a}{N} \right| \cdots \left| \frac{x-a}{n+1} \right| \\ &< \left| \frac{|x-a|^{N-1}}{(N-1)!} \right| q^{n-N+2} \end{aligned}$$

$\rightarrow 0$ as $n \rightarrow \infty$ thanks to $q < 1$.

In case of $a = 0$, the formula obtained in Taylor's theorem is known as *Maclaurin's formula* and the corresponding series that one obtains is known as *Maclaurin's series*.

Examples 4.0.5.

1. $f(x) = e^x$.

In this case $R_n(x) = \frac{x^{n+1}}{(n+1)!} f^{(n+1)}(c) = \frac{x^{n+1}}{(n+1)!} e^c = \frac{x^{n+1}}{(n+1)!} e^{\theta x}$, for some $\theta \in (0, 1)$.

Therefore for any given bounded interval I fixed,

$$|R_n(x)| = \left| \frac{x^{n+1}}{(n+1)!} \right| e^{\theta x} \leq C \frac{|x|^{n+1}}{(n+1)!} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Therefore

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \text{ (with convention of } 0! = 1).$$

2. $f(x) = \sin x$.

In this case it is easy to see that

$$|R_n(x)| \leq \frac{|x|^{2n+1}}{(2n+1)!} |\sin(c + \frac{n\pi}{2})| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Now use the fact that $|\sin x| \leq 1$ and follow as in example (i). Hence,

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

Definition 4.0.6. An infinitely differentiable function $f(x)$ is called *Real analytic* at $x = a$ if the function has Taylor series expansion: There exists $R > 0$ and

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n, \quad |x-a| < R.$$

Note that the series in the RHS may converge but not equal to $f(x)$. So to say that the function has its Taylor series we need to first check the remainder term $R_n(x)$ goes to zero as $n \rightarrow \infty$.

We state the following characterization of real analytic functions. The proof is beyond the scope of this course.

Theorem 4.0.7. The following are equivalent

1. $f(x)$ is real analytic at $x = a$
2. For every small interval I containing a , there exist constants $r > 0$ and $C > 0$ such that for all $k \in \mathbb{N} \cup \{0\}$:

$$|f^{(k)}(x)| \leq C \frac{k!}{r^k} \text{ for all } x \in I.$$

(2) $\implies$ (1): The remainder of Taylor's theorem R_n can be estimated as

$$\begin{aligned} |R_n(x)| &\leq \frac{C}{r^{n+1}} |x-a|^{n+1} \\ &= C \left(\frac{|x-a|}{r} \right)^{n+1} \rightarrow 0 \text{ if } |x-a| < r. \end{aligned}$$

As examples, we have already seen that $e^x, \sin x, \cos x$ satisfy the hypothesis (see the example above).

Example 4.0.8. Consider the function defined by

$$f(x) = \begin{cases} e^{-1/x}, & x > 0 \\ 0 & x \leq 0. \end{cases}$$

This function is infinitely differentiable and all its derivatives at 0 are equal to zero. So the Taylor series at zero is identically zero. But the function is not identically equal to zero in any interval containing zero.

5 Power series

Motivation: One of the motivations to study power series from the integration point of view is when we want to find the value of integral

$$\int_0^1 x^x dx$$

From our understanding of functions we can see that x^x is continuous in $[0, 1]$. We may write

$$\int_0^1 x^x dx = \int_0^1 e^{x \log x} dx$$

Now using the Taylor series of exponential function, we may write

$$e^{x \log x} = \sum_{n=0}^{\infty} \frac{x^n (\log x)^n}{n!}$$

Therefore the integral becomes

$$\int_0^1 e^{x \log x} dx = \int_0^1 \sum_{n=0}^{\infty} \frac{x^n (\log x)^n}{n!} dx$$

So IF WE CAN interchange the operation of integration and infinite sum then this is equal to

$$\int_0^1 e^{x \log x} dx = \sum_{n=0}^{\infty} \frac{1}{n!} \int_0^1 x^n (\log x)^n dx$$

This integral can be evaluated easily using integration by parts and one can get the value equal to

$$\int_0^1 x^x dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{n^n}$$

This is an alternating series and converges (using Dirichlet test.) In this lecture our aim is to

study the problem of the calculus of functions that are expressed as infinite series as above.

Motivation 2: Suppose we want to construct a continuous function that is equal to 0 for $x \leq 0$ and is equal to 1 for $x \geq 0$. Then we can simply fill the interval by straight line joining 0 and 1. Then we ask if we need the function to be differentiable also. Then we can try to construct a polynomial that satisfies the conditions of left and right derivatives to be zero at 0 and 1 and $f(0) = 0, f(1) = 1$. Such a polynomial is of degree minimum 3. This way, if we want m times differentiable function, then we can try a polynomial of very high degree say $2m$. Now we ask the question: Does there exist infinitely differentiable function? If there is one, then it cannot be any polynomial. The intuition suggests that we need a function with infinite degrees of freedom say $a_0 + a_1x + a_2x^2 + \dots + a_nx^n + \dots$ leading to the study of functions that are of infinite series. Indeed such a function exists and is of the exponential nature.

This raises a question of if we write a series $\sum a_nx^n$ then is this series finite always for any $x \neq 0$? The answer is NO. for example

Example 5.0.1. Consider the series $\sum_{n=0}^{\infty} (n!)x^n$. Then the n^{th} term $(n!)x^n$ does not approach zero for any $x \neq 0$. So this series is not finite for any $x \neq 0$.

The above example motivates one to study the set of points x for which an infinite series $\sum a_nx^n$ is finite.

Definition 5.0.2. Given a sequence of real numbers $\{a_n\}_{n=0}^{\infty}$, the series $\sum_{n=0}^{\infty} a_n(x-c)^n$ is called power series with center c .

It is easy to see that a power series converges for $x = c$. Power series is a function of x provided it converges for x . If a power series converges, then the domain of convergence is either a bounded interval or the whole of $\mathbb{R}$. So it is natural to study the largest interval where the power series converges.

Remark 5.1. If $\sum a_nx^n$ converges at $x = r$, then $\sum a_nx^n$ converges for $|x| < |r|$.

Proof. We can find $C > 0$ such that $|a_nr^n| \leq C$ for all n . Then

$$|a_nx^n| = |a_nr^n| \left| \frac{x}{r} \right|^n \leq C \left| \frac{x}{r} \right|^n.$$

Conclusion follows from comparison theorem and the fact that $q := \frac{|x|}{|r|} < 1$ and the series $\sum q^n$ is a geometric series.

Theorem 5.0.3. Consider the power series $\sum_{n=0}^{\infty} a_nx^n$. Suppose $\beta = \limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|}$ and $R = \frac{1}{\beta}$ (We define $R = 0$ if $\beta = \infty$ and $R = \infty$ if $\beta = 0$). Then

1. $\sum_{n=0}^{\infty} a_nx^n$ converges for $|x| < R$

2. $\sum_{n=0}^{\infty} a_n x^n$ diverges for $|x| > R$.

3. No conclusion if $|x| = R$.

Proof. Proof of (i) follows from the root test. For a proof, take $\alpha_n(x) = a_n x^n$ and $\alpha = \limsup \sqrt[n]{|\alpha_n|}$. For (ii), one can show that if $|x| > R$, then there exists a sub-sequence $\{a_n\}$ such that $a_n \not\rightarrow 0$. Notice that $\alpha = \beta|x|$. For (iii), observe as earlier that the series with $a_n = \frac{1}{n}$ and $b_n = \frac{1}{n^2}$ will have $R = 1$.

Similarly, we can prove:

Theorem 5.0.4. Consider the power series $\sum_{n=0}^{\infty} a_n x^n$. Suppose $\beta = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$ and $R = \frac{1}{\beta}$ (We define $R = 0$ if $\beta = \infty$ and $R = \infty$ if $\beta = 0$). Then

1. $\sum_{n=0}^{\infty} a_n x^n$ converges for $|x| < R$

2. $\sum_{n=0}^{\infty} a_n x^n$ diverges for $|x| > R$.

3. No conclusion if $|x| = R$.

Definition 5.0.5. The real number R in the above theorems is called the Radius of convergence of power series.

Examples 5.0.6. Find the interval of convergence of

$$(i) \sum \frac{x^n}{n} \quad (ii) \sum \frac{x^n}{n!} \quad (iii) \sum 2^{-n} x^{3n}$$

1. $\beta = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{n}{n+1} = 1$, and we know that the series does not converge for $x = 1$, but converges at $x = -1$. So the interval of convergence is $[-1, 1)$.

2. $\beta = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{1}{n+1} = 0$. Hence the series converges everywhere. The interval of convergence is $\mathbb{R}$.

3. To see the subsequent non-zero terms, we write the series as $\sum 2^{-n} (x^3)^n = \sum 2^{-n} y^n$. For this series $\beta_y = \limsup \sqrt[n]{|a_n|} = 2^{-1}$. Therefore, $\beta_x = 2^{-1/3}$ and radius of convergence $R = 2^{1/3}$. It diverges at $y = 2$ and at the other end point $y = -2$ it diverges (why?!).

Problem 7. Find the radius of convergence of the series $\sum a_n x^n$ where $a_n = \begin{cases} 2^n & n \text{ is even} \\ 2^{n-1} & n \text{ is odd} \end{cases}$

Here

$$\frac{a_{n+1}}{a_n} = \begin{cases} 4 & n \text{ even} \\ 1 & n \text{ odd} \end{cases}$$

Then $\limsup_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = 4$ and $\liminf_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$. Hence the limit of $\left| \frac{a_{n+1}}{a_n} \right|$ does not exist. But $\limsup_{n \rightarrow \infty} |a_n|^{\frac{1}{n}} = 2$ (check!). So the radius of convergence of the series is $\frac{1}{2}$.

Theorem 5.0.7. Continuity of power series: *The function f defined as power series*

$$f(x) = \sum a_n x^n, \quad |x| < R$$

is continuous at $x = 0$.

Proof. Proof follows from the following important estimate: For any $|x| < r < R$

$$|f(x) - a_0| \leq \sum_{n=1}^{\infty} |a_n| |x|^n \leq |x| \sum_{n=1}^{\infty} |a_n| |x|^{n-1} \leq |x| \sum_{n=1}^{\infty} |a_n| r^{n-1}$$

Since the power series converges, it is not difficult to show for any $r < R$ the series $\sum_{n=1}^{\infty} |a_n| r^{n-1}$ converges to S (say). Then

$$|f(x) - a_0| \leq |x| S \rightarrow 0 \text{ as } |x| \rightarrow 0.$$

Theorem 5.0.8. *Suppose $\sum a_n x^n = \sum b_n x^n$, $|x| < R$ then $a_n = b_n$ for all n .*

Proof. By continuity, taking $x \rightarrow 0$, we get $a_0 = b_0$. Now assume by induction $a_n = b_n$ for all $n = 0, 1, 2, \dots, k$. Then by trivial cancellation on both sides we get

$$a_{k+1} + a_{k+2}x + \dots = b_{k+1} + b_{k+2}x + \dots$$

then again by continuity, we get $a_{k+1} = b_{k+1}$.

The following theorem is very useful in identifying of some Taylor series.

Theorem 5.0.9. *(Term by term differentiation and integration): Suppose $f(x) = \sum_{n=0}^{\infty} a_n x^n$ converges for $|x| < R$. Then*

1. $\sum_{n=0}^{\infty} n a_n x^{n-1}$ converges in $|x| < R$ and is equal to $f'(x)$.
2. $\sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$ converges in $|x| < R$ and is equal to $\int_0^x f(t) dt$.

From this theorem one concludes that a power series is infinitely differentiable within its radius of convergence. Now it is natural to ask whether this series coincides with the Taylor series of the resultant function. The answer is yes and it is simple to prove that if $f(x) = \sum a_n x^n$, then $a_n = \frac{f^{(n)}}{n!}$. The above theorem is useful to find Taylor series of some functions.

Problem 8. Find the Taylor series of $f(x) = \tan^{-1} x$ about $x = 0$ and domain of its convergence.

Consider the function $\phi(x) = \frac{1}{1+x}$. Then by Taylor's theorem

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots (-1)^n x^n + (-1)^{n+1} \frac{x^{n+1}}{(1+c)^{n+2}}$$

Hence for $x \in (0, 1)$, we get $c > 0$ and the Remainder term

$$|R_n(x)| \leq \left| \frac{x^{n+1}}{(1+c)^{n+2}} \right| \leq |x|^{n+1} \rightarrow 0$$

On the other hand if we look for convergence of the Taylor series of $\frac{1}{1+x}$

$$\sum_{n=0}^{\infty} (-1)^n x^n$$

We can easily see that the radius of convergence is 1 and the series converges at $x = -1$ and diverges at $x = 1$. Hence the series converges in $(-1, 1]$. Now using the above theorem on term by term integration, we get

$$\tan^{-1} x = \int_0^x \frac{dx}{1+x^2} = \int_0^x (1 - x^2 + x^4 - \dots) dx = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots$$

It is easy to check that the Radius of convergence of this series is equal to 1 (Try!). At the end point $x = -1, 1$ this series is an alternating series that converges (check). Also, at the end point $x = 1$ we get interesting sum

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots = \tan^{-1}(1) = \frac{\pi}{4}.$$

Though the function $\tan^{-1} x$ is defined on all of $\mathbb{R}$, we see that the power series converges on $[-1, 1]$.

Example 5.0.10. Show that $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$ for all $x \in (-1, 1]$.

By integrating the series term by term $f(x) = \frac{1}{1+x}$. use the series expansion of $\frac{1}{1+x}$ for $|x| < 1$

and integrate it to obtain

$$\log(1+x) = \int_0^x \frac{1}{1+t} dt = \int_0^x [1 - t + t^2 - t^3 + \dots] = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$$

At the end points, one can use Abel's test to check the convergence of alternating series.

Proof of Theorem 5.0.9: Let $f_1(x) = \sum_{n=1}^{\infty} na_n x^{n-1}$, $|x| < R$ and since $f(x)$ converges for $|x| < R$, we can choose H such that $|x| + H < R$. Then for $|h| < H$ and $|x| < R$, we have

$$\begin{aligned} \left| \frac{f(x+h) - f(x)}{h} - f_1(x) \right| &= \left| \sum_{n=2}^{\infty} a_n \left\{ \frac{(x+h)^n - x^n - nhx^{n-1}}{h} \right\} \right| \\ &\leq |h| \left| \sum_{n=2}^{\infty} a_n \sum_{k=2}^n \binom{n}{k} x^{n-k} h^{k-2} \right| \\ &\leq |h| \sum_{n=2}^{\infty} |a_n| \sum_{k=2}^n \binom{n}{k} |x|^{n-k} H^{k-2} \\ &= \frac{|h|}{H^2} \sum_{n=2}^{\infty} |a_n| \sum_{k=2}^n \binom{n}{k} |x|^{n-k} H^k \\ &\leq \frac{|h|}{H^2} \sum_{n=0}^{\infty} |a_n| (|x| + H)^n = \frac{|h|}{H^2} M \end{aligned}$$

where $M = \sum_{n=0}^{\infty} |a_n| (|x| + H)^n$ which converges since $|x| + H < R$. Letting h goes to zero, we get

$$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = f_1(x) \quad \text{for } |x| < R$$

To prove the term by term integration formula, we define $F(x) = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$. Then apply (1) for F .

From this theorem one concludes that a power series is infinitely differentiable within its radius of convergence. We note that the power series may converge to a function on small interval, even though the function is defined on a much bigger interval. For example the function $\log(1+x)$ has power series that converges on $(-1, 1]$ but $\log(1+x)$ is defined on $(-1, \infty)$. This is due to the fact that the domain of convergence of power series is symmetric about the center.