Relationship Between Hypothesis Testing and Confidence Intervals

The Hypothesis Testing and Confidence Interval estimation are mathematically two sides of the exact same coin.

The Golden Rule: A $100(1-\alpha)\%$ confidence interval contains all the possible values of the population parameter for which the corresponding two-sided hypothesis test would *fail to reject* the null hypothesis at the α level of significance.

Conversely, if a specific null hypothesis value (e.g., θ_0) falls *outside* the $100(1-\alpha)\%$ confidence interval, a two-sided test will definitively *reject* that null hypothesis at significance level α.

Mathematical Proof of the Relationship Suppose we have an estimator $\hat{\theta}$ for an unknown parameter θ. Assume the standardized test statistic T follows a known symmetric distribution (like the Standard Normal Z or Student's t) under the null hypothesis.

The Hypothesis Test:
We wish to test $H_0 : \theta = \theta_0$ against $H_1 : \theta \neq \theta_0$ at significance level α. Our standardized test statistic is:

$$T_{test} = \frac{\hat{\theta} - \theta_0}{SE(\hat{\theta})}$$

where $SE(\hat{\theta})$ is the standard error of the estimator.

According to the decision rule for a two-sided test, we **fail to reject** H_0 if the test statistic falls within the critical values:

$$-t_{\alpha/2} \leq \frac{\hat{\theta} - \theta_0}{SE(\hat{\theta})} \leq t_{\alpha/2}$$

Algebraic Rearrangement:
Let us take the exact inequality that defines the "Fail to Reject" (Acceptance) region and isolate the hypothesized parameter θ_0 in the middle.

Multiply all parts by $SE(\hat{\theta})$:

$$-t_{\alpha/2} \cdot SE(\hat{\theta}) \leq \hat{\theta} - \theta_0 \leq t_{\alpha/2} \cdot SE(\hat{\theta})$$

Subtract $\hat{\theta}$ from all parts:

$$-\hat{\theta} - t_{\alpha/2} \cdot SE(\hat{\theta}) \leq -\theta_0 \leq -\hat{\theta} + t_{\alpha/2} \cdot SE(\hat{\theta})$$

Multiply by -1 (which flips the inequality signs):

$$\hat{\theta} + t_{\alpha/2} \cdot SE(\hat{\theta}) \geq \theta_0 \geq \hat{\theta} - t_{\alpha/2} \cdot SE(\hat{\theta})$$

Rearranging from smallest to largest:

$$\hat{\theta} - t_{\alpha/2} \cdot SE(\hat{\theta}) \leq \theta_0 \leq \hat{\theta} + t_{\alpha/2} \cdot SE(\hat{\theta})$$

Conclusion:
The boundaries of this inequality are exactly the lower and upper bounds of the $100(1-\alpha)\%$ Confidence Interval for θ. Therefore, the mathematical condition for failing to reject H_0 is completely identical to the mathematical condition for θ_0 being located inside the Confidence Interval.

[Testing a Normal Mean (Known Variance)] We can observe this duality directly in the foundational case of testing the mean of a normal population where the true variance, σ^2, is known.

Suppose we draw a random sample of size n and want to test if the true population mean μ equals a specific baseline value μ_0:

$$H_0 : \mu = \mu_0 \quad \text{vs.} \quad H_1 : \mu \neq \mu_0$$

using a significance level of α.

The standard Z-statistic for this test is constructed by standardizing the sample mean $\bar{X}$:

$$Z_{test} = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}$$

We fail to reject H_0 if $-z_{\alpha/2} \leq Z_{test} \leq z_{\alpha/2}$.

Simultaneously, the standard formula for the $100(1-\alpha)\%$ Confidence Interval for the population mean μ is:

$$\left[\bar{X} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}}, \quad \bar{X} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right]$$

The Practical Interpretation: Instead of calculating the Z-statistic manually and comparing it to a critical threshold from the standard normal table, one can simply compute the Confidence Interval.

- If the hypothesized mean μ_0 falls **inside** the calculated interval, the mathematical equivalence guarantees that $|Z_{test}| \leq z_{\alpha/2}$. You immediately **Fail to Reject** H_0 at the α significance level.
- If the hypothesized mean μ_0 falls **outside** the interval, it mathematically guarantees that $|Z_{test}| > z_{\alpha/2}$. You immediately **Reject** H_0 at the α significance level.

Summary of Advantages

Why do modern statisticians often prefer reporting Confidence Intervals over pure p-values or standard Hypothesis Tests?

1. **Incorporated Testing:** As proven above, the CI already acts as a hypothesis test. You can test any null hypothesis simply by checking if the value falls inside the interval.
2. **Direction and Magnitude:** A p-value only tells you that a difference exists (statistical significance). A CI tells you the estimated size of that difference and its direction (practical significance).
3. **Precision:** The width of the CI provides an immediate visual and numerical measure of the estimate's precision. A very

wide interval indicates high uncertainty (likely due to a small sample size or high variance), which a standalone p-value obscures.

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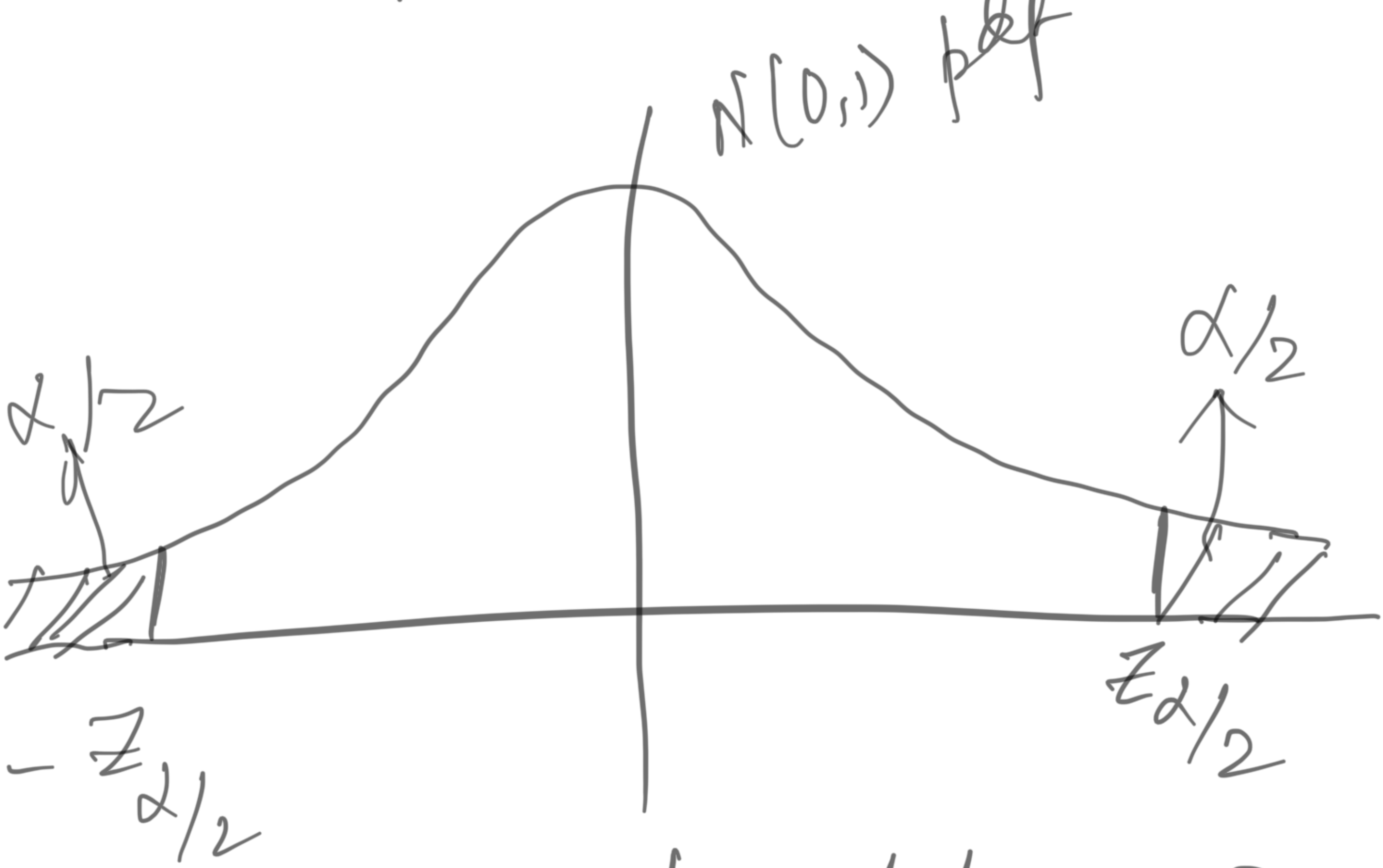
Relation between Hypothesis testing and Confidence interval

Suppose $X_1, \ldots, X_n$ is a random sample from $N(\mu, \sigma^2)$ pop. with σ^2 known.

① We want to find $(1-\alpha)$ confidence interval for μ. Pivotal quantity

$\bar{X} - \mu \quad \sim N(0,1)$

$$T = \frac{\qquad}{\sigma/\sqrt{n}}$$



From table/calculator we find $z_{\alpha/2}$, such that for $T \sim N(0,1)$

$$\mathbb{P}_{\sigma}\left(T \in [-z_{\alpha/2}, z_{\alpha/2}] \right) = 1-\alpha$$

$$\mathbb{P}\left(|T| < z_{\alpha/2} \right) = 1 - \alpha$$

$$\Rightarrow \mathbb{P}\left(\bar{X} - \frac{\sigma}{\sqrt{n}} z_{\alpha/2} \le \mu \le \bar{X} + \frac{\sigma}{\sqrt{n}} z_{\alpha/2} \right)$$

$$= 1 - \alpha$$

So, the CI is

$$\left[\bar{X} - \frac{\sigma}{\sqrt{n}} z_{\alpha/2} , \bar{X} + \frac{\sigma}{\sqrt{n}} z_{\alpha/2} \right]$$

(II) Hypothesis testing

$H_0 : \mu = \mu_0$, μ_0 is known

vs

$H_1 : \mu \neq \mu_0$

the test statistic is

$$T = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} \sim N(0,1)$$

At α level of significance

we reject H_0 at level α

if

$$C = \left\{ |T| > Z_{\alpha/2} \right\}$$

$$= \left\{ \left| \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} \right| > Z_{\alpha/2} \right\}$$

The acceptance region is

$$\Omega \setminus C$$

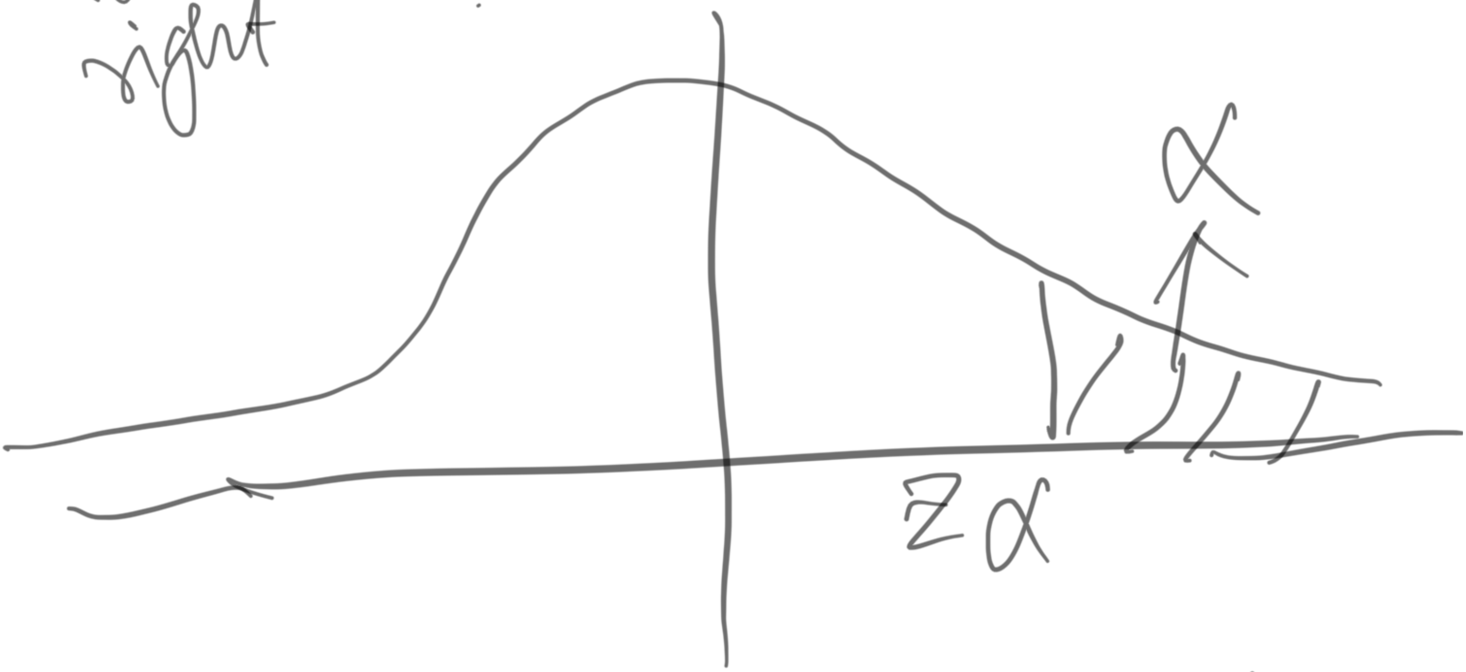
$$= \left\{ \left| \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} \right| \le z_{\alpha/2} \right\}$$

$$= \left\{ \bar{X} - \frac{\sigma}{\sqrt{n}} z_{\alpha/2} \le \mu_0 \le \bar{X} + \frac{\sigma}{\sqrt{n}} z_{\alpha/2} \right\}$$

$$= \left\{ \mu_0 \in \underbrace{\left[\bar{X} - \frac{\sigma}{\sqrt{n}} z_{\alpha/2}, \bar{X} + \frac{\sigma}{\sqrt{n}} z_{\alpha/2} \right]}_{CI} \right\}$$

$$\Rightarrow C = \left\{ \mu_0 \notin \left[\bar{X} - \frac{\sigma}{\sqrt{n}} z_{\alpha/2}, \bar{X} + \frac{\sigma}{\sqrt{n}} z_{\alpha/2} \right] \right\}$$

~~one~~ sided
right



$$P(T < Z_\alpha) = 1 - \alpha$$

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optional

(Y_i, X_i), $i = 1, 2, \ldots, n$

dependent var   Independent var.

Model

$$Y_i = \beta_0 + \beta_1 X_i + \varepsilon_i$$

where

① ε_i's are IID with zero mean and var σ^2.

② X_i's are non-random

Estimation of β_0 & β_1

$$\varepsilon_i = (Y_i - \beta_0 - \beta_1 X_i)$$

Square error loss

$$S(\beta_0, \beta_1) = \sum_{i=1}^{n} \varepsilon_i^2$$

$$= \sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x_i)^2$$

Using OLS, we get

$$\hat{\beta}_0 = b_0 \quad , \quad \hat{\beta}_1 = b_1$$

Predicted y_i,

$$\hat{y}_i = \underline{b_0} + \underline{b_1}\, x_i$$

$$Y_i = \underbrace{\beta_0 + \beta_1 X_i}_{} + \varepsilon_i$$

ith residual

$$Y_i - y_i =$$