# MTL108: Solution to Problem Set-2

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**Problem 1:** There are $k$ people in a room. Assume each person's birthday is equally likely to be any of the 365 days of the year (we exclude February 29), and that people's birthdays are independent (we assume there are no twins in the room). What is the probability that two or more people in the group have the same birthday?

**Solution.** Let the event $D$ be "at least two people share a birthday". It is easier to compute the complement event $D^c =$ "all $k$ birthdays are distinct". Under the uniform independent model,

$$\mathbb{P}(D^c) = \frac{365 \cdot 364 \cdot 363 \cdots (365 - k + 1)}{365^k} = \prod_{i=0}^{k-1} \frac{365 - i}{365},$$

provided $k \leq 365$. Hence

$$\mathbb{P}(D) = 1 - \prod_{i=0}^{k-1} \frac{365 - i}{365}.$$

If $k > 365$, the pigeonhole principle forces a repeat and so $\mathbb{P}(D) = 1$.

This formula is the usual form of the birthday-problem probability. For example, at $k = 23$ the probability exceeds $1/2$ (this is a classical surprising fact).

**Problem 2:** Suppose we choose a positive integer at random, according to some unknown probability distribution. Suppose we know that $\mathbb{P}(\{1, 2, 3, 4, 5\}) = 0.3$, $\mathbb{P}(\{4, 5, 6\}) = 0.4$ and $\mathbb{P}(\{1\}) = 0.1$. What are the largest and smallest possible values of $\mathbb{P}(\{2\})$?

**Solution.** Let $p_i = \mathbb{P}(\{i\})$ for $i = 1, 2, \ldots$. The given information becomes

$$p_1 + p_2 + p_3 + p_4 + p_5 = 0.3, \qquad p_4 + p_5 + p_6 = 0.4, \qquad p_1 = 0.1.$$

From the first and third equations we get

$$p_2 + p_3 + p_4 + p_5 = 0.2.$$

Hence

$$p_2 = 0.2 - (p_3 + p_4 + p_5).$$

Since probabilities are nonnegative, $p_3, p_4, p_5 \geq 0$, so the sum $p_3 + p_4 + p_5$ ranges between 0 and (at most) 0.2. Therefore:

$$\boxed{0 \leq p_2 \leq 0.2.}$$

*Attainability (constructive examples).*

- To attain the maximum $p_2 = 0.2$, set $p_3 = p_4 = p_5 = 0$. Then $p_1 = 0.1, p_2 = 0.2$. The equation $p_4 + p_5 + p_6 = 0.4$ forces $p_6 = 0.4$. The remaining probability mass (if any) can be assigned to $p_7, p_8, \ldots$ so that the total sums to 1 (for instance assign the remainder 0.3 to $p_7$). This yields a valid probability distribution.
- To attain the minimum $p_2 = 0$, set $p_2 = 0$ and choose $p_3, p_4, p_5$ so that $p_3 + p_4 + p_5 = 0.2$; for example $p_3 = 0.2, p_4 = p_5 = 0$. Then $p_6$ must be 0.4 (since $p_4 + p_5 + p_6 = 0.4$). Again put any remaining mass on higher indices to reach total 1. This yields a valid distribution with $p_2 = 0$.

Thus the largest possible value is 0.2 and the smallest possible value is 0.

**Problem 3:** Show that for any three events $A$, $B$ and $C$,

$$\mathbb{P}(A \cup B \cup C) = \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) - \mathbb{P}(A \cap B) - \mathbb{P}(A \cap C) - \mathbb{P}(B \cap C) + \mathbb{P}(A \cap B \cap C).$$

**Solution.** This is the inclusion–exclusion formula for three sets. There are several proofs; one clear combinatorial/probabilistic proof is by adding and subtracting overlaps so that every elementary outcome that belongs to at least one of $A, B, C$ is counted exactly once on the right-hand side. We discussed this in a lecture; a different proof is provided below.

Start from the two-set identity

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B),$$

and apply it twice. Consider

$$\mathbb{P}(A \cup B \cup C) = \mathbb{P}((A \cup B) \cup C) = \mathbb{P}(A \cup B) + \mathbb{P}(C) - \mathbb{P}((A \cup B) \cap C).$$

Now substitute the two-set formula for $\mathbb{P}(A \cup B)$ and expand

$$(A \cup B) \cap C = (A \cap C) \cup \mathbb{P}(B \cap C),$$

then apply the two-set formula again to $\mathbb{P}((A \cap C) \cup \mathbb{P}(B \cap C))$. Carrying out these substitutions and simplifications yields exactly

$$\mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) - \mathbb{P}(A \cap B) - \mathbb{P}(A \cap C) - \mathbb{P}(B \cap C) + \mathbb{P}(A \cap B \cap C).$$

This completes the proof.

**Problem 4:** (*Inclusion–Exclusion Principle for $n$ Events*). Let $A_1, \ldots, A_n$ be $n$ events. Prove the inclusion–exclusion formula:

$$\mathbb{P}\Big(\bigcup_{i=1}^{n} A_i\Big) = \sum_{1 \le i \le n} \mathbb{P}(A_i) - \sum_{1 \le i < j \le n} \mathbb{P}(A_i \cap A_j) + \sum_{1 \le i < j < k \le n} \mathbb{P}(A_i \cap A_j \cap A_k) - \cdots + (-1)^{n+1} \mathbb{P}\Big(\bigcap_{i=1}^{n} A_i\Big).$$

**Solution (proof by induction).** We prove the formula by induction on $n$.

Base case $n = 2$: we know for two sets,

$$\begin{aligned}\mathbb{P}(A_1 \cup A_2) &= \mathbb{P}(A_1) + \mathbb{P}(A_2) - \mathbb{P}(A_1 \cap A_2)\\ &= \sum_{i\in\{1,2\}} \mathbb{P}(A_i) - \sum_{1\le i<j\le 2} \mathbb{P}(A_i \cap A_j).\end{aligned}$$

So, the hypothesis holds for $n = 2$.

Inductive step: assume the formula holds for $n-1$ events. So, if we denote $B = \bigcup_{i=1}^{n-1} A_i$, then we have

$$\mathbb{P}(B) = \sum_{i=1}^{n-1} \mathbb{P}(A_i) - \sum_{1\le i<j\le n-1} \mathbb{P}(A_i \cap A_j) + \cdots + (-1)^n\, \mathbb{P}\Big(\bigcap_{i=1}^{n-1} A_i\Big). \tag{1}$$

Next, using $\mathbb{P}(C \cup D) = \mathbb{P}(C) + \mathbb{P}(D) - \mathbb{P}(C \cap D)$ and $\bigcup_{i=1}^{n} A_i = B \cup A_n$, we have

$$\mathbb{P}\Big(\bigcup_{i=1}^{n} A_i\Big) = \mathbb{P}(B \cup A_n) = \mathbb{P}(B) + \mathbb{P}(A_n) - \mathbb{P}(B \cap A_n).$$

Substituting $\mathbb{P}(B)$ from the inductive hypothesis (1) and using

$$B \cap A_n = \bigcup_{i=1}^{n-1} (A_i \cap A_n),$$

we have

$$\begin{aligned}\mathbb{P}\Big(\bigcup_{i=1}^{n} A_i\Big) &= \mathbb{P}(B) + \mathbb{P}(A_n) - \mathbb{P}(B \cap A_n)\\ &= \left[\sum_{i=1}^{n-1} \mathbb{P}(A_i) - \sum_{1\le i<j\le n-1} \mathbb{P}(A_i \cap A_j) + \cdots + (-1)^n\, \mathbb{P}\Big(\bigcap_{i=1}^{n-1} A_i\Big)\right]\\ &\quad + \mathbb{P}(A_n) - \mathbb{P}\left(\bigcup_{i=1}^{n-1} (A_i \cap A_n)\right)\end{aligned}$$

Applying the inclusion–exclusion formula (for $n-1$ sets) to the union $\bigcup_{i=1}^{n-1} (A_i \cap A_n)$ using (1), we have

$$\mathbb{P}\left(\bigcup_{i=1}^{n-1} (A_i \cap A_n)\right) = \sum_{i=1}^{n-1} \mathbb{P}(A_i \cap A_n) - \sum_{1\le i<j\le n-1} \mathbb{P}(A_i \cap A_n \cap A_j) + \cdots + (-1)^n\, \mathbb{P}\Big(\bigcap_{i=1}^{n-1} A_i \cap A_n\Big).$$

Substituting in the above expression and simplifying term-by-term gives exactly the $n$-term alternating sum in the statement. Also the combinatorial signs match since each $k$-fold intersection containing $A_n$ appears with opposite signs and cancels appropriately. Thus the formula holds for $n$. This completes the induction.

An alternative direct proof is combinatorial, proof sketch: for each elementary outcome $\omega$, count how many times it is counted by the right-hand side according to the number $m$ of the $A_i$'s that contain $\omega$; the alternating sum evaluates to 1 whenever $m \ge 1$ and to 0 when $m = 0$, so the RHS equals the indicator of the union and integrating gives the formula.

**Problem 5:** (*Boole's Inequality*). Let $A_1, \dots, A_n$ be events. Show

$$\mathbb{P}\Big(\bigcup_{i=1}^{n} A_i\Big) \le \sum_{i=1}^{n} \mathbb{P}(A_i).$$

**Solution.** For $n = 2$ we have

$$\mathbb{P}(A_1 \cup A_2) = \mathbb{P}(A_1) + \mathbb{P}(A_2) - \mathbb{P}(A_1 \cap A_2),$$

and $\mathbb{P}(A_1 \cap A_2) \ge 0$, so

$$\mathbb{P}(A_1 \cup A_2) \le \mathbb{P}(A_1) + \mathbb{P}(A_2),$$

Assume it holds for $n - 1$. Then

$$\mathbb{P}\Big(\bigcup_{i=1}^{n} A_i\Big) = \mathbb{P}\Big(\bigcup_{i=1}^{n-1} A_i\Big) + \mathbb{P}(A_n) - \mathbb{P}\Big(\Big(\bigcup_{i=1}^{n-1} A_i\Big) \cap A_n\Big).$$

Since the last term is nonnegative we get

$$\mathbb{P}\Big(\bigcup_{i=1}^{n} A_i\Big) \le \mathbb{P}\Big(\bigcup_{i=1}^{n-1} A_i\Big) + \mathbb{P}(A_n) \le \sum_{i=1}^{n-1} \mathbb{P}(A_i) + \mathbb{P}(A_n),$$

where the final inequality uses the induction hypothesis. This proves the inequality for $n$.

This inequality is sometimes called the *union bound* and is extremely useful because it gives a simple (though not always tight) upper bound for the probability of a union.



**Problem 6:** Two fair dice are rolled. Let $X$ and $Y$ be the outcome of the first die and the second die, respectively. Determine which of the following statements is/are true:

(a) $\mathbb{P}(X + Y = \text{ a prime number }) = \dfrac{5}{12}$,

(b) $\mathbb{P}(|X - Y| = \text{ a prime number }) = \dfrac{4}{9}$,

(c) $\mathbb{P}(X + Y = \text{ a perfect square}) = \dfrac{1}{6}$,

(d) $\mathbb{P}(|X - Y| = \text{ a prime number }) = \dfrac{5}{9}$.

**Solution.** There are 36 equally likely ordered outcomes $(x, y)$ with $x, y \in \{1, \dots, 6\}$.

**(a)** Sums possible are $2, \dots, 12$. Prime sums in this range are $2, 3, 5, 7, 11$. Count ordered outcomes giving these sums:

$$\begin{aligned}
&\text{sum } 2: \ (1,1) \ \Rightarrow 1,\\
&\text{sum } 3: \ (1,2),(2,1) \ \Rightarrow 2,\\
&\text{sum } 5: \ (1,4),(2,3),(3,2),(4,1) \ \Rightarrow 4,\\
&\text{sum } 7: \ (1,6),(2,5),(3,4),(4,3),(5,2),(6,1) \ \Rightarrow 6,\\
&\text{sum } 11: \ (5,6),(6,5) \ \Rightarrow 2.
\end{aligned}$$

Total $1 + 2 + 4 + 6 + 2 = 15$ favourable outcomes, so

$$\mathbb{P}(\text{sum is prime}) = \frac{15}{36} = \frac{5}{12}.$$

Thus (a) is **true**.

**(b)** Consider $|X - Y|$. Possible absolute differences are $0, 1, 2, 3, 4, 5$. Primes among these are $2, 3, 5$ (note 1 is not prime). Count ordered outcomes:

- $|X - Y| = 2$: pairs $(1, 3), (2, 4), (3, 5), (4, 6)$ and their reverses $\Rightarrow$ 8 outcomes.
- $|X - Y| = 3$: $(1, 4), (2, 5), (3, 6)$ and reverses $\Rightarrow$ 6 outcomes.
- $|X - Y| = 5$: $(1, 6)$ and $(6, 1)$ $\Rightarrow$ 2 outcomes.

Total favourable $8 + 6 + 2 = 16$, hence

$$\mathbb{P}(|X - Y| \text{ is prime}) = \frac{16}{36} = \frac{4}{9}.$$

So (b) is **true**.

**(c)** Perfect-square sums in $[2, 12]$ are 4 and 9.

- sum 4: $(1, 3), (2, 2), (3, 1) \Rightarrow$ 3 outcomes.
- sum 9: $(3, 6), (4, 5), (5, 4), (6, 3) \Rightarrow$ 4 outcomes.

Total 7 outcomes, so probability $7/36 \neq 1/6$ (since $1/6 = 6/36$). Thus (c) is **false**.

**(d)** This repeats (b) but with $5/9$; since (b) gave $4/9$, (d) is **false**.

**Problem 7:** Three biscuit-making machines A, B and C: A makes 35%, B makes 27%, C makes the rest. Broken rates: A: 4%, B: 1%, C: 9%. Selected biscuit is broken. What is the probability it was *NOT* made by A?

**Solution.** Let events $A, B, C$ denote the biscuit came from that machine; let $E$ denote "broken". Then events $A, B, C$ denote a partition of the sample space, as a biscuit is made by exactly one of the machines. With this modeling, we have:

$$\mathbb{P}(A) = 0.35, \ \mathbb{P}(B) = 0.27, \ \mathbb{P}(C) = 1 - 0.35 - 0.27 = 0.38.$$

Also, given the conditional probabilities are

$$\mathbb{P}(E|A) = 0.04, \ \mathbb{P}(E|B) = 0.01, \ \mathbb{P}(E|C) = 0.09.$$

We are interested in finding, $\mathbb{P}(A^c|\, E)$. We know that

$$\mathbb{P}(A^c|\, E) + \mathbb{P}(A|\, E) = 1 \quad \Rightarrow \mathbb{P}(A^c|\, E) = 1 - \mathbb{P}(A|\, E).$$

Now we aim to find $\mathbb{P}(A|\, E)$.

Using total probability, the probability that a randomly selected biscuit is broken is given by

$$\mathbb{P}(E) = \mathbb{P}(A)\mathbb{P}(E|A) + \mathbb{P}(B)\mathbb{P}(E|B) + \mathbb{P}(C)\mathbb{P}(E|C).$$

Substituting values, we have

$$\mathbb{P}(E) = 0.35(0.04) + 0.27(0.01) + 0.38(0.09) = 0.014 + 0.0027 + 0.0342 = 0.0509.$$

Now by Bayes' rule,

$$\mathbb{P}(A \mid E) = \frac{\mathbb{P}(A)\mathbb{P}(E \mid A)}{\mathbb{P}(E)} = \frac{0.014}{0.0509} = \frac{140}{509} \approx 0.27505.$$

Hence the probability it was *NOT* made by A is

$$\mathbb{P}(A^c | E) = 1 - \mathbb{P}(A \mid E) = 1 - \frac{140}{509} = \frac{369}{509} \approx 0.72495.$$

**Problem 8:** A fair die is rolled twice independently. Let $X, Y$ be the two outcomes. Define $Z = X + Y$ and $W$ the remainder when $Z$ is divided (integer division) by 6 (so $W \in \{0, 1, 2, 3, 4, 5\}$). Prove or disprove:

(i) Events $\{X = a\}$ and $\{W = b\}$ are independent for $a = 4, 5, 6$ and $b = 0, 1, 2$,

(ii) Events $\{X = a\}$ and $\{W = b\}$ are independent for $a = 1$, $b = 5$,

(iii) Events $\{X = a\}$ and $\{Z = b\}$ are independent for $a = 1$, $b = 1$,

(iv) Events $\{X = a\}$ and $\{Z = b\}$ are independent for $a = 1$, $b = 5$.

**Solution.** Outcome table is given below.

| | | X | | | | | |
|---|---|---|---|---|---|---|---|
| | Z | 1 | 2 | 3 | 4 | 5 | 6 |
| Y | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
| | 6 | 7 | 8 | 9 | 10 | 11 | 12 |

| | | X | | | | | |
|---|---|---|---|---|---|---|---|
| | W | 1 | 2 | 3 | 4 | 5 | 6 |
| Y | 1 | 2 | 3 | 4 | 5 | 0 | 1 |
| | 2 | 3 | 4 | 5 | 0 | 1 | 2 |
| | 3 | 4 | 5 | 0 | 1 | 2 | 3 |
| | 4 | 5 | 0 | 1 | 2 | 3 | 4 |
| | 5 | 0 | 1 | 2 | 3 | 4 | 5 |
| | 6 | 1 | 2 | 3 | 4 | 5 | 0 |

First note $\mathbb{P}(X = a) = 1/6$ for any $a \in \{1, \ldots, 6\}$. Compute $\mathbb{P}(W = b)$: observe that sums $Z$ range $2, \ldots, 12$ with frequency $1, 2, 3, 4, 5, 6, 5, 4, 3, 2, 1$ respectively. Grouping sums by remainder when divided by 6: each remainder class $b \in \{0, 1, 2, 3, 4, 5\}$ is attained by exactly two sums whose frequencies sum to 6. Hence

$$\mathbb{P}(W = b) = \frac{6}{36} = \frac{1}{6} \quad \text{for all } b \in \{0, \ldots, 5\}.$$

Thus $W$ is uniform on $\{0, \ldots, 5\}$.

Now for fixed $a, b$,

$$\{X = a\} \cap \{W = b\} = \{X = a,\ (a+Y) \equiv b \pmod 6\} = \{X = a,\ Y \equiv b - a \pmod 6\}.$$

For each remainder class when divided by 6 there is exactly one element of $\{1, \ldots, 6\}$ having that remainder when divided by 6. Thus for any given $a, b$ there is exactly one value $y \in \{1, \ldots, 6\}$ meeting $y \equiv b - a \pmod 6$. Therefore

$$\mathbb{P}(\{X = a\} \cap \{W = b\}) = \frac{1}{36}.$$

But $\mathbb{P}(X = a)\mathbb{P}(W = b) = (1/6)(1/6) = 1/36$. Hence for *all* $a \in \{1, \ldots, 6\}$ and all $b \in \{0, \ldots, 5\}$ the events $\{X = a\}$ and $\{W = b\}$ are independent. In particular:

(i) For $a = 4, 5, 6$ and $b = 0, 1, 2$ independence holds. ✓

(ii) For $a = 1, b = 5$ independence also holds (by the general argument). ✓

Next consider independence with $Z$ (the actual sum).

For (iii): $b = 1$ is impossible since $Z \geq 2$, so $\mathbb{P}(Z = 1) = 0$. Then for any $a$,

$$\mathbb{P}(\{X = a\} \cap \{Z = 1\}) = 0 = \mathbb{P}(X = a)\mathbb{P}(Z = 1),$$

so $\{X = a\}$ and $\{Z = 1\}$ are trivially independent. In particular for $a = 1, b = 1$ they are independent. ✓

For (iv): take $a = 1, b = 5$. Compute

$$\mathbb{P}(Z = 5) = \frac{\#\{(x, y) : x + y = 5\}}{36} = \frac{4}{36} = \frac{1}{9},$$

and

$$\mathbb{P}(\{X = 1\} \cap \{Z = 5\}) = \mathbb{P}(X = 1, Y = 4) = \frac{1}{36}.$$

But $\mathbb{P}(X = 1)\mathbb{P}(Z = 5) = (1/6)(1/9) = \frac{1}{54} \neq \frac{1}{36}$. Therefore $\{X = 1\}$ and $\{Z = 5\}$ are *not* independent. ×

**Problem 9:** Let $A$ and $B$ be events with $\mathbb{P}(A) > 0$. Prove or disprove

$$\mathbb{P}(B \mid A) \geq 1 + \frac{\mathbb{P}(B)}{\mathbb{P}(A)} - \frac{1}{\mathbb{P}(A)}.$$

**Solution.** Rearranging the right-hand side,

$$1 + \frac{\mathbb{P}(B)}{\mathbb{P}(A)} - \frac{1}{\mathbb{P}(A)} = \frac{\mathbb{P}(A) + \mathbb{P}(B) - 1}{\mathbb{P}(A)}.$$

So the inequality is equivalent to

$$\frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)} \geq \frac{\mathbb{P}(A) + \mathbb{P}(B) - 1}{\mathbb{P}(A)},$$

because

$$\mathbb{P}(A \cap B) \geq \mathbb{P}(A) + \mathbb{P}(B) - 1.$$

But this last inequality is always true: from

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) \leq 1$$

we obtain

$$\mathbb{P}(A \cap B) \geq \mathbb{P}(A) + \mathbb{P}(B) - 1.$$

Thus the claimed inequality holds for all events with $\mathbb{P}(A) > 0$. Equality holds exactly when $\mathbb{P}(A \cup B) = 1$ (i.e. the union has probability one).