MTL108: Solution to Problem Set-1

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Problem 1: (*Discussion type*) A box contains 3 identical red balls and 2 identical blue balls. Two balls are drawn without replacement. A student uses naive probability, assuming all pairs are equally likely (ignoring order), and calculates the probability of getting two red balls as $\frac{3}{5} \cdot \frac{2}{4}$. Compute the probability using the combinatorial approach and compare with the student's approach.

Solution. Using the combinatorial approach (unordered pairs): the number of ways to choose 2 balls from the 3 red ones is

$$\binom{3}{2} = 3,$$

and the total number of ways to choose 2 balls from all 5 balls is

$$\binom{5}{2} = 10.$$

Thus the probability of drawing two red balls is

$$\mathbb{P}(\text{two red}) = \frac{\binom{3}{2}}{\binom{5}{2}} = \frac{3}{10}.$$

Now compute the student's sequential product:

$$\frac{3}{5} \cdot \frac{2}{4} = \frac{6}{20} = \frac{3}{10}.$$

Comparison and comment. Both methods give the same numerical result, $\frac{3}{10}$. This is because the sequential calculation correctly accounts for the conditional probability of the second draw given the first (without replacement). The combinatorial count directly counts unordered outcomes; the sequential product counts ordered draws but the arithmetic is equivalent after simplification. Both approaches are valid here.

Problem 2: In a survey, 30 students reported whether they liked their tea, coffee, or soft-drink. 15 liked tea, 20 liked coffee, and 9 liked soft-drink. Additionally, 12 students liked both tea and coffee, 5 liked coffee and soft-drink, 6 liked tea and soft-drink, and 3 liked all three. How many students dislike all three? Explain why your answer is correct.

Solution. Let T, C, S denote the sets of students who like tea, coffee, and soft-drink respectively. Using the inclusion–exclusion principle for three sets,

$$|T \cup C \cup S| = |T| + |C| + |S| - |T \cap C| - |T \cap S| - |C \cap S| + |T \cap C \cap S|.$$

Substitute the given numbers:

$$\begin{aligned} |T \cup C \cup S| &= 15 + 20 + 9 - 12 - 6 - 5 + 3 \\ &= 24. \end{aligned}$$

So 24 students like at least one of the three beverages. Since there are 30 students total, the number who dislike all three is

$$30 - 24 = 6.$$

Why this is correct. Inclusion–exclusion guarantees exact counting when overlaps among the sets are known; we counted singles, subtracted double-counted pairs, then added back the triple-counted ones. Hence 6 students like none of the three.

Problem 3: Consider a biased coin with sample space $\Omega = \{H, T\}$ and σ-algebra $\mathcal{C} = 2^\Omega$, the power set of Ω. The probability measure is defined as $P(\{H\}) = 0.7$, $P(\{T\}) = 0.3$. Verify that P satisfies the axioms of a probability measure. Then, compute the probability of the event $\{H, T\}$.

Solution. A probability measure P on a measurable space $(\Omega, \mathcal{C})$ must satisfy the following axioms:

(*i*) *Non-negativity:* $P(A) \geq 0$ for all $A \in \mathcal{C}$.

(*ii*) *Normalization:* $P(\Omega) = 1$.

(*iii*) *Countable additivity:* If $A_1, A_2, \dots \in \mathcal{C}$ are disjoint, then

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i).$$

Step 1: Non-negativity. We are given $P(\{H\}) = 0.7 \geq 0$, $P(\{T\}) = 0.3 \geq 0$. Also, $P(\emptyset) = 0$ by convention. Hence non-negativity holds.

Step 2: Normalization. Since $\Omega = \{H, T\}$, we have

$$P(\Omega) = P(\{H, T\}) = P(\{H\}) + P(\{T\}) = 0.7 + 0.3 = 1.$$

So the normalization axiom holds.

Step 3: Countable additivity. The σ-algebra is finite, so it suffices to check finite additivity. For disjoint sets $A, B \in \mathcal{C}$, we have

$$P(A \cup B) = P(A) + P(B).$$

For instance, if $A = \{H\}, B = \{T\}$, then

$$P(A \cup B) = P(\{H, T\}) = 1, \quad P(A) + P(B) = 0.7 + 0.3 = 1.$$

Thus additivity is satisfied, and hence countable additivity holds.

Step 4: Probability of the event $\{H, T\}$. By Step 2 we already computed:

$$P(\{H, T\}) = 1.$$

Conclusion: The given P satisfies all three axioms of a probability measure. The probability of the whole sample space event $\{H, T\}$ is 1.

Problem 4: Consider two fair six-sided dice rolled. Define the probability space $(\Omega, \mathcal{C}, P)$, where Ω is the set of all ordered pairs (i, j), $i, j \in \{1, 2, \ldots, 6\}$. Specify $\mathcal{C}$ as the power set and define P. Calculate the probability of the event $A = \{(i, j) \mid i + j \geq 10\}$.

Solution. A standard probability space for two fair dice is:

$$\Omega = \{(i, j) \mid i, j \in \{1, \ldots, 6\}\}, \qquad \mathcal{C} = 2^{\Omega},$$

and since the dice are fair and independent, each elementary outcome (i, j) is assigned probability

$$P(\{(i, j)\}) = \frac{1}{36}.$$

For any event $E \subseteq \Omega$, $P(E) = \dfrac{|E|}{36}$.

Now compute $A = \{(i, j) : i + j \geq 10\}$. The possible sums ≥ 10 are $10, 11, 12$. Count ordered outcomes:

$$\begin{aligned} &\text{sum } 10 : (4, 6), (5, 5), (6, 4) \quad (3 \text{ outcomes}), \\ &\text{sum } 11 : (5, 6), (6, 5) \quad (2 \text{ outcomes}), \\ &\text{sum } 12 : (6, 6) \quad (1 \text{ outcome}). \end{aligned}$$

So $|A| = 3 + 2 + 1 = 6$. Hence

$$P(A) = \frac{6}{36} = \frac{1}{6}.$$

Problem 5: Let $\Omega = \{1, 2, 3, 4\}$ and $\mathcal{C} = \{\emptyset, \{1, 2\}, \{3, 4\}, \{1, 2, 3, 4\}\}$. Check if $\mathcal{C}$ is a σ-algebra by testing all three axioms. If it fails, identify which axiom is violated and construct a minimal σ-algebra containing $\{1, 2\}$.

Solution. Recall the three axioms for a collection $\mathcal{C}$ to be a σ-algebra on Ω:

(a) $\Omega \in \mathcal{C}$.

(b) If $A \in \mathcal{C}$ then $A^c \in \mathcal{C}$.

(c) $\mathcal{C}$ is closed under countable unions (equivalently finite unions here, since Ω is finite).

Check them:

- Axiom 1: $\Omega = \{1, 2, 3, 4\} \in \mathcal{C}$. ✓
- Axiom 2 (complements): $\emptyset^c = \Omega \in \mathcal{C}$. Also $\{1, 2\}^c = \{3, 4\} \in \mathcal{C}$ and $\{3, 4\}^c = \{1, 2\} \in \mathcal{C}$. ✓
- Axiom 3 (countable unions): The union of any subcollection of $\mathcal{C}$ is one of $\emptyset, \{1, 2\}, \{3, 4\}, \Omega$, because

$$\{1, 2\} \cup \{3, 4\} = \Omega, \quad \{1, 2\} \cup \emptyset = \{1, 2\}, \text{ etc.}$$

So $\mathcal{C}$ is closed under unions. ✓

Therefore $\mathcal{C}$ satisfies all three axioms and is a σ-algebra.

Minimal σ-algebra containing $\{1, 2\}$*.* The minimal σ-algebra that contains $\{1, 2\}$ must also contain its complement $\{3, 4\}$, $\emptyset$, and Ω. Thus the minimal σ-algebra is exactly

$$\{\emptyset, \{1, 2\}, \{3, 4\}, \Omega\},$$

which is already the given collection.

Problem 6: (*Discussion type*) In a naive attempt to find the probability that a randomly chosen point in a unit square $[0, 1]\times[0, 1]$ lies within the circle inscribed in it (radius 0.5, center $(0.5, 0.5)$), a student assumes points are equally likely and estimates the probability as the ratio of diameters. Do you agree with his/her approach? Explain your reasoning in either case.

Solution. We interpret "randomly chosen point in the unit square" in the usual geometric-probability sense as chosen according to the uniform distribution on the square (i.e. uniform with respect to area). Under that interpretation the probability that the point lies inside the inscribed circle equals the ratio of the area of the circle to the area of the square.

The circle has radius $r = 0.5$, so its area is $\pi r^2 = \pi(0.5)^2 = \frac{\pi}{4}$. The square has area 1. Thus the probability is

$$\mathbb{P}(\text{point in circle}) = \frac{\text{area of circle}}{\text{area of square}} = \frac{\pi/4}{1} = \frac{\pi}{4} \approx 0.785398\ldots$$

Why the diameter-ratio is wrong. The student who takes the ratio of diameters is implicitly using a one-dimensional measure (length) instead of the correct two-dimensional measure (area). The diameter of the circle is 1, the side of the square is 1 too, so the ratio of diameters would be $1/1 = 1$ (or if they compared radii $0.5/0.5 = 1$), which is incorrect here. Possibly the student meant to take the ratio of diameters of the inscribed circle to the diagonal of the square or some other 1D quantity; regardless any approach that uses ratios of lengths instead of areas is not appropriate for a 2D uniform distribution. The correct measure is area, giving $\pi/4$.

Thus, the student's diameter-based approach is incorrect, as it ignores the part of the square outside the circle. Assuming uniform distribution over the square in terms of area, the correct probability is $\pi/4$.

Problem 7: A probability space has $\Omega = \{a, b, c, d\}$, $\mathcal{C} = 2^{\Omega}$, and probability measure $P(\{a\}) = 0.2$, $P(\{b\}) = 0.3$, $P(\{c\}) = 0.4$, $P(\{d\}) = 0.1$. Compute the probabilities of the events $A = \{a, b\}$, $B = \{b, c\}$, and their union $A \cup B$. Verify that the probability measure satisfies $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Solution. Since singleton events are disjoint and probabilities add on disjoint unions:

$$P(A) = P(\{a, b\}) = P(\{a\}) + P(\{b\}) = 0.2 + 0.3 = 0.5,$$

$$P(B) = P(\{b, c\}) = P(\{b\}) + P(\{c\}) = 0.3 + 0.4 = 0.7,$$

$$A \cup B = \{a, b, c\} \quad \Rightarrow \quad P(A \cup B) = 0.2 + 0.3 + 0.4 = 0.9,$$

$$A \cap B = \{b\} \quad \Rightarrow \quad P(A \cap B) = 0.3.$$

Check the inclusion–exclusion identity:

$$P(A) + P(B) - P(A \cap B) = 0.5 + 0.7 - 0.3 = 0.9 = P(A \cup B).$$

So the identity holds.

Problem 8: For any two events A and B,

$$\mathbb{P}(A) + \mathbb{P}(B) - 1 \leq \mathbb{P}(A \cap B) \leq \mathbb{P}(A \cup B) \leq \mathbb{P}(A) + \mathbb{P}(B).$$

Proof.

- From the basic identity

  $$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B),$$

  and since $\mathbb{P}(A \cup B) \leq 1$, rearrange to get

  $$\mathbb{P}(A \cap B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cup B) \geq \mathbb{P}(A) + \mathbb{P}(B) - 1.$$

  This proves the leftmost inequality.
- Trivially $A \cap B \subseteq A \cup B$, so $\mathbb{P}(A \cap B) \leq \mathbb{P}(A \cup B)$. This is the middle inequality.
- Also $A \subseteq A \cup B$ and $B \subseteq A \cup B$, so by subadditivity (or directly from the identity above, since $\mathbb{P}(A \cap B) \geq 0$),

  $$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) \leq \mathbb{P}(A) + \mathbb{P}(B).$$

  This proves the rightmost inequality.

When equalities occur?

(a) $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$ iff $\mathbb{P}(A \cap B) = 0$. In other words, the events are disjoint.

(b) $\mathbb{P}(A \cap B) = \mathbb{P}(A \cup B)$ iff $A = B$ (up to events of probability zero). If $A = B$, then intersection equals union and the probabilities coincide.

(c) $\mathbb{P}(A \cap B) = \mathbb{P}(A) + \mathbb{P}(B) - 1$ iff $\mathbb{P}(A \cup B) = 1$. Equivalently, the union covers the whole sample space (almost surely).

These provide simple, natural set-theoretic criteria for when each inequality becomes an equality.

Problem 9: Prove that an algebra may not be a σ-algebra.

Proof. A counterexample is given by Example 1 in Topic 2.