

**Lecture notes
Calculus**

Chapter-5

Multiple integrals

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1 Double integrals

Let $f(x, y)$ be a real valued function defined over a domain $\Omega \subset \mathbb{R}^2$. To start with, let us assume that Ω be the rectangle $R = (a, b) \times (c, d)$. We partition the rectangle with node points (x_k, y_k) , where

$$a = x_1 < x_2 < \dots, x_n = b, \text{ and } c = y_1 < y_2 < \dots, y_n = d.$$

Let R_{nm} be the small sub-rectangle with above vertices. Now we can define Upper and lower Riemann sum as

$$U(P_n, f) = \sum_{n,m} \sup_{R_{nm}} f(x, y) |R_{nm}|$$

and

$$L(P_n, f) = \sum_{n,m} \inf_{R_{nm}} f(x, y) |R_{nm}|$$

where $|R_{nm}|$ is the area of the rectangle R_{nm} . Here we may define the norm of partition P_n as $\|P_n\| = \max_i \sqrt{|x_i - x_{i-1}|^2 + |y_i - y_{i-1}|^2}$. Then by our understanding of definite integral we can define the upper, lower integrals and $f(x, y)$ is integrable if and only if

$$\inf\{U(P, f) : P\} = \sup\{L(P, f) : P\}.$$

The definite integral is defined as

$$\iint_{\Omega} f(x, y) dA = \inf\{U(P, f) : P\} = \sup\{L(P, f) : P\}.$$

It can be shown that $\iint_{\Omega} f(x, y) dA = \lim_{\|P_n\| \rightarrow 0} S(P_n, f)$, where $S(P_n, f) = \sum_{n,m} f(x_k, y_k) |R_{nm}|$.

We have the following **Fubini's theorem** for rectangle:

Suppose $f(x, y)$ is integrable over $R = (a, b) \times (c, d)$, then

$$\iint_R f(x, y) dA = \int_a^b \left(\int_c^d f(x, y) dy \right) dx = \int_c^d \left(\int_a^b f(x, y) dx \right) dy$$

In case of $f(x, y) \geq 0$ we may interpret this as the volume of the solid formed by the surface $z = f(x, y)$ over the rectangle R . This is precisely the "sum" of areas of the cross section $A(x) = \int_c^d f(x, y) dy$ between $x = a$ and $x = b$. Since x varies over all of (a, b) , this sum is nothing but the integral $\int_a^b A(x) dx$.

For any general bounded domain Ω , we can divide the domain into small sub domains Ω_k and consider the upper, lower sum exactly as above by replacing R_{nm} by Ω_k . Then a function

$f(x, y) : \Omega \rightarrow \mathbb{R}$ is integrable if the supremum of lower sums and infimum of upper sums exist and are equal. We may define

$$\iint_{\Omega} f(x, y) dA = \lim_{\|P_n\| \rightarrow 0} \sum_k f(x_k, y_k) |\Omega_k|$$

The **basic properties** of the definite integral like integrability of $f \pm g$, kf and domain decomposition theorems holds in this case also.

We call a domain as **y -regular** if $\Omega = \{(x, y) : a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$ for some continuous functions g_1, g_2 . Similarly, Ω is **x -regular** if $\Omega = \{(x, y) : c \leq y \leq d, h_1(y) \leq x \leq h_2(y)\}$. A domain is called regular if it is either x -regular or y -regular. Then we have the following Fubini's theorem for regular domains.

Theorem 1.0.1 *Let $f(x, y)$ be continuous over Ω .*

1. *If Ω is y -regular, i.e., $\Omega = \{(x, y) : a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$. Then*

$$\iint_{\Omega} f(x, y) dA = \int_a^b \left(\int_{g_1(x)}^{g_2(x)} f(x, y) dy \right) dx.$$

2. *If Ω is x -regular, i.e., $\Omega = \{(x, y) : c \leq y \leq d, h_1(y) \leq x \leq h_2(y)\}$. Then*

$$\iint_{\Omega} f(x, y) dA = \int_c^d \left(\int_{h_1(y)}^{h_2(y)} f(x, y) dx \right) dy.$$

This theorem basically says that if a function is integrable over a domain Ω , then the value of integral is does not depend on order of integration. That is we can integrate with respect to x first followed by y or vice versa.

Example 1.0.1 *Evaluate the integral $\int_{\Omega} (x + y + xy) dA$ where Ω is the triangle bounded by $y = 0, x = 1$ and $y = x$.*

Solution: The triangle is regular in both x and y . The given triangle is

$$\{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq x\} = \{(x, y) : 0 \leq y \leq x, y \leq x \leq 1\}$$

Therefore, taking it as y -regular we see that the domain is bounded below by $y = g_1(x) = 0$ and above by $y = g_2(x) = x$ over $x = 0$ to $x = 1$. Hence,

$$\begin{aligned} \iint_{\Omega} (x + y + xy) dA &= \int_{x=0}^{x=1} \left(\int_{y=0}^x (x + y + xy) dy \right) dx \\ &= \int_0^1 \left(x^2 + \frac{x^2}{2} + \frac{x^3}{2} \right) dx = \frac{15}{24} \end{aligned}$$

Similarly, taking it as x -regular, we see that the domain is bounded below by $x = h_1(y) = y$ and above by $x = h_2(y) = 1$ over $y = 0$ to 1 . Hence

$$\iint_{\Omega} (x + y + xy) dA = \int_{y=0}^1 \left(\int_{x=y}^1 (x + y + xy) dx \right) dy = \frac{15}{24}$$

Example 1.0.2 Evaluate the integral $\iint_{\Omega} (2+4x) dA$ where Ω is the domain bounded by $y = x$ and $y = x^2$.

Solution:

$$\begin{aligned} \iint_{\Omega} (2+4x) dA &= \int_{x=0}^1 \left(\int_{y=x^2}^x (2+4x) dy \right) dx \\ &= \int_0^1 (2x + 2x^2 - 4x^3) dx = 2/3 \end{aligned}$$

On the other hand, this is also equal to $\int_{y=0}^1 \left(\int_{x=y}^{\sqrt{y}} (2+4x) dx \right) dy$.

Remark 1.0.1 1. When $f(x, y) = 1$, then we approximate the area of Ω as $A \sim \sum_k \Omega_k = \sum_k f(x_k, y_k) |\Omega_k|$ where $f = 1$. By the definition of Riemann integral this sum converges to $\iint_{\Omega} dA$ as $\|P_n\| \rightarrow 0$.

2. As discussed in the beginning, when $f(x, y) \geq 0$, the $\iint_{\Omega} f(x, y) dA$ is the volume of the solid bounded above by $z = f(x, y)$ and below by Ω .

Problem 1.0.1 Find the area bounded by $y = 2x^2$ and $y^2 = 4x$.

Solution: The two parabola's intersect at $(0, 0)$ and $(1, 1)$. Hence the area is

$$A = \int_0^1 \int_{2x^2}^{2\sqrt{x}} dy dx = \int_0^1 (2\sqrt{x} - 2x^2) dx = \frac{2}{3}.$$

Problem 1.0.2 Find the volume of the solid under the paraboloid $z = x^2 + y^2$ over the bounded domain R bounded by $y = x$, $x = 0$ and $x + y = 2$.

Solution: The domain of integration R is Y -regular bounded above by $x + y = 2$ and below by $y = x$ with x varying over $(0, 1)$.

$$\begin{aligned} V &= \iint_R (x^2 + y^2) dA = \int_0^1 \left(\int_{y=x}^{y=2-x} (x^2 + y^2) dy \right) dx \\ &= \int_0^1 \left. \frac{y^3}{3} + yx^2 \right|_{y=x}^{y=2-x} dx = \int_0^1 \left(\frac{1}{3}((2-x)^3 - x^3) + x^2(2-2x) \right) dx \end{aligned}$$

Problem 1.0.3 Find the volume of the solid bounded above by the surface $z = x^2$ and below by the plane region R bounded by the parabola $y = 2 - x^2$, $y = x$.

Solution: The points of intersection of $y = x, y = 2 - x^2$ are $x = -2, 1$. So $R = \{(x, y) : -2 \leq x \leq 1, x \leq y \leq 2 - x^2\}$. Therefore,

$$V = \int_{x=-2}^1 \int_{y=x}^{2-x^2} x^2 dy dx = \int_{-2}^1 x^2(2 - x^2 - x) dx$$

Change of order

Consider the evaluation of integral $\iint_R \frac{\sin x}{x} dA$ over the triangle formed by $y = 0, x = 1$ and $y = x$. Since the can be extended as continuous function over R , by the basic properties of Riemann integral the function is integrable. Now by Fubini's theorem, the value of integral does not depend on the order of integration. As we noted earlier R is regular in x and y . If we take it as x regular, then $R = \{(x, y) : 0 \leq y \leq 1, y \leq x \leq 1\}$ and try to evaluate the integral, then

$$\iint_R \frac{\sin x}{x} dA = \int_0^1 \left(\int_{x=y}^1 \frac{\sin x}{x} dx \right) dy.$$

This is singular integral and difficult to evaluate.

But when we consider R to be y -regular, we see that $R = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq x\}$. Then the given integral is

$$\iint_R \frac{\sin x}{x} dA = \int_0^1 \int_{y=0}^x \frac{\sin x}{x} dy dx = \int_0^1 \sin x dx = 1 - \cos 1$$

At times this technique can be used to evaluate some complicated definite integrals, for example,

Problem 1.0.4 Evaluate the integral $\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x} dx$, $a, b > 0$.

Solution: This integral is equivalent to

$$\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x} dx = \int_0^\infty \left(\int_a^b e^{-xy} dy \right) dx$$

The domain of integration is the infinite strip $\{(x, y) : 0 \leq x \leq \infty, a \leq y \leq b\}$. Changing the order of integration, we get

$$\begin{aligned} \int_0^\infty \left(\int_a^b e^{-xy} dy \right) dx &= \int_{y=a}^b \left(\int_0^\infty e^{-xy} dx \right) dy \\ &= \ln \frac{b}{a} \end{aligned}$$

1.1 Double integrals in Polar form

Suppose we are given a bounded region whose boundaries are given by polar equations, say $f_1(r, \theta) = 0, f_2(r, \theta) = 0$. Then we divide the region into smaller "polar rectangles" whose sides have constant r, θ values.

Suppose $f(r, \theta)$ is defined over a region R defined using the polar equations, $R : \alpha \leq \theta \leq \beta, g_1(\theta) \leq r \leq g_2(\theta)$. Then we divide the r range by $\Delta r, 2\Delta r, \dots, m\Delta r$ and $\alpha, \alpha + \Delta\theta, \dots, \alpha + m'\Delta\theta = \beta$. Let ΔA be the polar rectangle with sides $r_k - \Delta r/2, r_k + \Delta r/2$ and $\alpha + k\Delta\theta, \alpha + (k+1)\Delta\theta$. Then we define the Riemann sum as

$$S_n = \sum_k f(r_k, \theta_k) \Delta A_k.$$

The area of small "polar rectangle" A_k is

$$\Delta A_k = \text{area of outer sector} - \text{area of inner sector} = r_k \Delta r \Delta \theta.$$

As $\|P_n\| \rightarrow 0$, we get

$$S_n = \sum_k f(r_k, \theta) r_k \Delta r \Delta \theta \rightarrow \iint_R f(r, \theta) r dr d\theta.$$

Problem 1.1.1 Find the area common to the cardioids $r = 1 + \cos \theta$ and $r = 1 - \cos \theta$.

Solution: Since the region is symmetric with respect to x -axis and y -axis, the required area is

$$\begin{aligned} A &= 4 \int_{\theta=0}^{\pi/2} \int_{r=0}^{1-\cos \theta} r dr d\theta \\ &= 4 \int_0^{\pi/2} \frac{1}{2} (1 - 2\cos \theta + \cos^2 \theta) d\theta = 2 \left(\frac{\pi}{2} - 2 + \int_0^{\pi/2} \cos^2 \theta d\theta \right) \end{aligned}$$

Problem 1.1.2 Evaluate $\iint_R 3y dA$ where R is the region bounded below by x -axis and above by the cardioid $r = 1 - \cos \theta$.

Solution: The given integral is equivalent to

$$\iint_R 3y dA = \int_{\theta=0}^{\pi} \int_{r=0}^{1-\cos \theta} r \sin \theta r dr d\theta.$$

Problem 1.1.3 Evaluate $I = \int_0^{\infty} e^{-x^2} dx$.

Solution: Recall that $2I = \Gamma(\frac{1}{2})$. Using the Fubini's theorem, we may write

$$I^2 = \left(\int_0^{\infty} e^{-x^2} dx \right) \left(\int_0^{\infty} e^{-y^2} dy \right) = \int_0^{\infty} \int_0^{\infty} e^{-(x^2+y^2)} dA$$

Where the integration is over the first quadrant $(0, \infty) \times (0, \infty)$. So representing this in polar form we integrate over $\{(r, \theta) : 0 \leq r < \infty, 0 \leq \theta \leq \frac{\pi}{2}\}$. Therefore, the above integral becomes

$$\int_0^{\pi/2} \int_0^{\infty} e^{-r^2} r dr d\theta = \frac{\pi}{4}.$$

2 Triple (Volume) integrals

Let $f(x, y, z)$ be a real valued function defined over a closed and bounded region of space $\mathbb{R}^3$. For example the solid ball or rectangular box. Now we want to define the definite integral of $f(x, y, z)$ over such regions.

We partition the region by small planes parallel to the coordinate axes. Then we obtain small rectangular cubes over which the function will be approximated by $f(x_k, y_k, z_k)$. We form the Riemann sum

$$S_n = \sum_k f(x_k, y_k, z_k) |\Omega_k|,$$

where $|\Omega_k|$ is the volume of the small rectangle. Now by our understanding of Riemann sums we choose refinement of partitions in such way that $\max_k |\Omega_k| \rightarrow 0$. Then we obtain the definite integral as

$$\iiint_{\Omega} f(x, y, z) dV = \lim_{n \rightarrow \infty} S_n.$$

Evaluation of integrals in three dimensions is done again using Fubini's theorem. In this case again Fubini's theorem states

Theorem 2.0.1 Suppose $f(x, y, z)$ is integrable over $\Omega \subset \mathbb{R}^3$, then

$$\begin{aligned} \iiint_{\Omega} f(x, y, z) dV &= \int_x \int_y \int_z f(x, y, z) dz dy dx = \int_x \int_z \int_y f(x, y, z) dx dz dy \\ &= \int_z \int_x \int_y f(x, y, z) dy dx dz = \int_z \int_y \int_x f(x, y, z) dx dy dz \\ &= \int_y \int_x \int_z f(x, y, z) dz dx dy = \int_y \int_z \int_x f(x, y, z) dy dz dx \end{aligned}$$

To evaluate the triple integrals we follow the following steps:

1. Draw a line parallel to z axis that passes through the point (x, y) of R where R is the projection of Ω onto $\mathbb{R}^2$.
2. Identify the upper surface and lower surface through which this line passes at most once.
3. Identify the upper curve and lower curve of the projection R and limits of integration.

It is easy to see from the definition, the volume of Ω is

$$V = \lim_{k \rightarrow \infty} \sum_k |\Omega_k| = \lim_{k \rightarrow \infty} \sum_k 1 |\Omega_k| = \iiint_{\Omega} 1 dV$$

Problem 2.0.1 Find the volume of the region bounded by the surfaces $z = x^2 + 3y^2$ and $z = 8 - x^2 - y^2$.

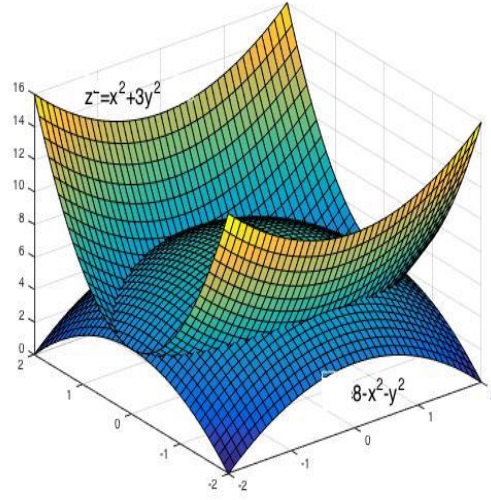


Figure 1: Volume bounded by the surfaces

Solution: The volume is $V = \iiint_{\Omega} dz dy dx$, where Ω is bounded above by the surface $z = 8 - x^2 - y^2$ and below by the surface $z = x^2 + 3y^2$. Therefore, the limits of z are from $z = x^2 + 3y^2$ to $z = 8 - x^2 - y^2$.

The Projection of Ω on xy -plane is the solution of

$$8 - x^2 - y^2 = x^2 + 3y^2 \implies x^2 + 2y^2 = 4.$$

Therefore the limits of x and y are to be determined by $R : x^2 + 2y^2 = 4$. Hence

$$\begin{aligned} V &= \iint_R \int_{y=x^2+3y^2}^{8-x^2-y^2} dz dA \\ &= \int_{-2}^2 \int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} (8 - 2x^2 - 4y^2) dy dx \\ &= \int_{-2}^2 \left((8 - x^2)y - \frac{4}{3}y^3 \right) \Big|_{y=-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} \\ &= \frac{4\sqrt{2}}{3} \int_{-2}^2 (4 - x^2)^{3/2} dx = 8\pi\sqrt{2}. \end{aligned}$$

Problem 2.0.2 Find the volume of the region bounded by $x + z = 1$, $y + 2z = 2$ in the first quadrant.

Solution: Draw line parallel to z -axis and note that the upper surfaces are: $2z + y = 2$ over triangle bounded by $x = 0, y = 1, y = 2x$ and $z = 1 - x$ over the triangle bounded by

$y = 0, x = 1, y = 2x$. Therefore,

$$V = \int_{y=0}^2 \int_{x=0}^{y/2} \int_{z=0}^{\frac{2-y}{2}} dz \, dx \, dy + \int_{x=0}^1 \int_{y=0}^{2x} \int_{z=0}^{1-x} dz \, dy \, dx$$

On the other hand, by first drawing the line parallel to x -axis, we get

$$V = \int_{z=0}^1 \int_{y=0}^{2-2z} \int_{x=0}^{1-z} dx \, dy \, dz$$

Taking the line parallel to y -axis we get

$$V = \int_{x=0}^1 \int_{z=0}^{1-x} \int_{y=0}^{2-2z} dy \, dz \, dx$$

Example 2.0.1 (Order of integration) Evaluate $\int_{z=0}^4 \int_{y=0}^1 \int_{x=2y}^2 \frac{2 \cos(x^2)}{\sqrt{z}} dx \, dy \, dz$.

Note that the projection of Ω onto xy -plane is the triangle bounded by $y = 0, x = 2$ and $x = 2y$. So changing the order of integration in x and y , we get

$$\begin{aligned} I &= \int_{z=0}^4 \int_{x=0}^2 \int_{y=0}^{x/2} \frac{2 \cos(x^2)}{\sqrt{z}} dy \, dx \, dz. \\ &= \int_{z=0}^4 \int_{x=0}^2 \frac{x \cos(x^2)}{\sqrt{z}} dx \, dz = 2 \sin 4. \end{aligned}$$

2.1 Substitutions in multiple integrals

Suppose a domain G in uv -plane is transformed onto a domain Ω of xy -plane by a transformation $x = g(u, v), y = h(u, v)$. Then any function of x, y may be written as a function of u, v . Then the relation between the double integral over G and Ω is

$$\iint_{\Omega} f(x, y) dx \, dy = \iint_G f(g(u, v), h(u, v)) |J(u, v)| du \, dv$$

where J is the Jacobian given by

$$J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$$

The main idea of the proof is as follows. Let $(u, v), (u + \Delta u, v), (u + \Delta u, v + \Delta v)$ and $(u, v + \Delta v)$ be the vertices of the rectangle in the uv -plane. Let ΔA_k be its area element. Under the transformation this points are mapped to $(x_1, y_1) = (g(u, v), h(u, v)), (x_2, y_2) = (g(u + \Delta u, v), h(u + \Delta u, v)), (x_3, y_3) = (g(u + \Delta u, v + \Delta v))$ and $(x_4, y_4) = (g(u, v + \Delta v), h(u, v + \Delta v))$. Then by Taylor's theorem

$$g(u + \Delta u, v) = g(u, v) + \frac{\partial g}{\partial u} \Delta u + o((\Delta u)^2)$$

$$g(u + \Delta u, v + \Delta v) = g(u, v) + \frac{\partial g}{\partial u} \Delta u + \frac{\partial g}{\partial v} \Delta v + o((\Delta u)^2) + o((\Delta v)^2)$$

Then the area of the "rectangle" in xy -plane $\Delta \tilde{A}_k$ is

$$\begin{aligned} \Delta \tilde{A}_k &\approx |(x_3 - x_1)(y_3 - y_1) - (x_3 - x_2)(y_3 - y_2)| \\ &\approx |J| \Delta u \Delta v + o((\Delta u)^2) + o((\Delta v)^2) \end{aligned}$$

Taking this as the area in the Riemann sum of $f(x, y)$ we get the required formula.

Example 2.1.1 Evaluate the integral $I = \int_0^4 \int_{y/2}^{1+\frac{y}{2}} \frac{2x-y}{2} dx dy$.

Solution: The domain of integration is a parallelogram with vertices $(0, 0)$, $(1, 0)$, $(3, 4)$ and $(2, 4)$. One has to divide the domain into 3 domains. Instead we can take the transformation $u = \frac{2x-y}{2}$, $v = \frac{y}{2}$. Then the inverse transformation is $x = u + v$, $y = 2v$. Then

$$J = \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = 2.$$

Under this transformation, the parallelogram is transformed into cube with vertices $(0, 0)$, $(1, 0)$, $(1, 2)$ and $(0, 2)$. Now by change of variable formula

$$I = \iint f(u + v, 2v) 2 du dv = \int_0^2 \int_0^1 2u du dv = 2.$$

Example 2.1.2 Evaluate the integral $I = \int_0^1 \int_0^{1-x} \sqrt{x+y} (y-2x)^2 dA$.

Solution: The given domain is the triangle bounded by $x = 0$, $y = 0$ and $x + y = 1$. In this case the integrand is complicated....so we can take transformation $u = x + y$ and $v = y - 2x$. Under this transformation, the given triangle will be transformed into triangle bounded by $v = u$, $v = -2u$ and $u = 1$. The inverse of this transformation is $x = \frac{u-v}{3}$ and $y = \frac{2u+v}{3}$. Hence the Jacobian

$$J = \begin{vmatrix} 1/3 & -1/3 \\ 2/3 & 1/3 \end{vmatrix} = 1/3.$$

Hence

$$I = \int_0^1 \int_{v=-2u}^u \sqrt{uv}^2 dv du$$

Example 2.1.3 Evaluate the integral $I = \iint_R \frac{dA}{(2-x^2-y^2)^2}$ over $R : x^2 + y^2 \leq 1$.

Solution: Taking the transformation $x = r \cos \theta$, $y = r \sin \theta$, we get

$$J = \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix} = r.$$

By substitution formula,

$$I = \int_0^{2\pi} \int_0^1 \frac{r \, dr \, d\theta}{(2-r^2)^2} = 2\pi \int_1^2 \frac{dt}{2t^2} = \pi/2.$$

Substitution formula for triple integrals

As discussed above suppose a three dimensional domain G is transformed onto a domain D with a transformation $x = x(u, v, w), y = y(u, v, w), z = z(u, v, w)$, then

$$\iiint_D f(x, y, z) dV = \iiint_G F(u, v, w) |J(u, v, w)| dV$$

where

$$J = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix}.$$

Example 2.1.4 Evaluate $\iiint_{\Omega} (x^2y + 3xyz) dV$ where $R = \{(x, y, z) : 1 \leq x \leq 2, 0 \leq xy \leq 2, 0 \leq z \leq 1\}$.

solution: We take the transformation $u = x, v = xy$ and $w = z$. Then the planes $x = 1, 2$ transforms to $u = 1, 2$. The plane $y = 0$ transforms to $v = 0$. The surface $xy = 2$ transforms to $v = 2$. Then the Jacobian J is

$$\frac{1}{J} = \begin{vmatrix} 1 & 0 & 0 \\ y & x & 0 \\ 0 & 0 & 1 \end{vmatrix} = x = u.$$

Now by substitution formula,

$$\begin{aligned} I &= \int_{u=1}^2 \int_{v=0}^2 \int_{w=0}^1 (uv + 3vw) \frac{1}{u} dw \, dv \, du \\ &= \int_1^2 \int_0^2 (v + \frac{3v}{2u}) dv \, du \\ &= \int_1^2 (2 + \frac{3}{u}) du = 2 + 3 \ln 2. \end{aligned}$$

Cylindrical coordinates: A point P in the space ($\mathbb{R}^3$) is represented by (r, θ, z) where r, θ are polar coordinates of the projection of P on to xy -plane and z is the z distance of the projection from P . When we take the transformation $x = r \cos \theta, y = r \sin \theta, z = z$, the Jacobian is

$$J = \begin{vmatrix} x_r & x_\theta & x_z \\ y_r & y_\theta & y_z \\ z_r & z_\theta & z_z \end{vmatrix} = \begin{vmatrix} \cos \theta & -r \sin \theta & 0 \\ \sin \theta & r \cos \theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r$$

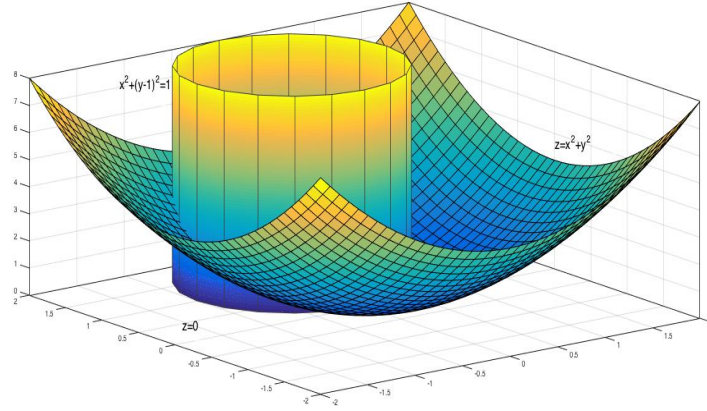


Figure 2: Volume bounded.

Example 2.1.5 Find the volume of the cylinder $x^2 + (y - 1)^2 = 1$ bounded by $z = x^2 + y^2$ and $z = 0$.

Solution: Drawing a line parallel to z axis, we see that the limits of z are from 0 to $x^2 + y^2$ and the projection onto xy -plane is the disc: $R : x^2 + (y - 1)^2 \leq 1$. Therefore,

$$V = \iint_R \int_{z=0}^{x^2+y^2} dz dA$$

Now taking the cylindrical coordinates $x = r \cos \theta, y = r \sin \theta, z = z$ we get the projection to be

$$r^2 \cos^2 \theta + r^2 \sin^2 \theta - 2r \sin \theta = 0$$

$$\text{i.e., } r(r - 2 \sin \theta) = 0 \implies r = 0 \text{ to } r = 2 \sin \theta$$

$$\begin{aligned} V &= \int_{\theta=0}^{\pi} \int_{r=0}^{2 \sin \theta} \int_{z=0}^{r^2} r dz dr d\theta \\ &= \int_{\theta=0}^{\pi} \int_{r=0}^{2 \sin \theta} r^3 dr d\theta \\ &= 4 \int_0^{\pi} \sin^4 \theta d\theta = \frac{5\pi}{4} \end{aligned}$$

Spherical polar coordinates: A point P in the space is represented by (ρ, θ, ϕ) where ρ is the distance of P from the origin, ϕ is the angle made by the ray OP with positive z axis and θ is the angle made by the projection of P (onto xy -plane) with positive x -axis. So it is not difficult to see that the relation with cartesian coordinates: The projection of P on xy -plane has polar representation: $x = r \cos \theta, y = r \sin \theta$ where r is the distance of the projected point to origin. Therefore $r = \rho \sin \phi$. From the definition of ϕ it is easy to see that $z = \rho \cos \phi$ and

$$x = \rho \sin \phi \cos \theta, y = \rho \sin \phi \sin \theta, z = \rho \cos \phi$$

The Jacobian in this case is

$$J = \begin{vmatrix} x_\rho & x_\theta & x_\phi \\ y_\rho & y_\theta & y_\phi \\ z_\rho & z_\theta & z_\phi \end{vmatrix} = \begin{vmatrix} \sin \phi \cos \theta & -\sin \phi \sin \theta & \cos \phi \cos \theta \\ \rho \sin \phi \sin \theta & \rho \cos \theta \sin \phi & 0 \\ \cos \phi & 0 & -\rho \sin \phi \end{vmatrix} = -\rho^2 \sin \phi$$

To find limits of integration in spherical coordinates,

1. Draw a ray from the origin to find the surfaces $\rho = g(\theta, \phi)$, $\rho = g_2(\theta, \phi)$ where it enters the region and leaves the region.
2. Rotate this ray away and towards z -axis to find the limits of ϕ
3. Identify the projection R of the domain on the xy -plane and polar form of R to write the limits of θ .

Example 2.1.6 Evaluate $\iiint_{\Omega} \frac{dV}{\sqrt{1+x^2+y^2+z^2}}$ where Ω is the unit ball $x^2+y^2+z^2 \leq 1$.

Solution: Going to spherical polar coordinates $x = \rho \sin \phi \cos \theta$, $y = \rho \sin \phi \sin \theta$, $z = \rho \cos \phi$, we get

$$\begin{aligned} I &= \int_{\rho=0}^1 \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi} \frac{\rho^2}{\sqrt{1+\rho^2}} \sin \phi \, d\phi \, d\theta \, d\rho \\ &= (2\pi \times 2) \int_0^1 \frac{\rho^2}{\sqrt{1+\rho^2}} d\rho = 4\pi(\sqrt{2} - \frac{1}{2} \ln(\sqrt{2}+1)). \end{aligned}$$

Example 2.1.7 Evaluate $I = \iiint_{\Omega} x dV$ where Ω is the part of the ball $x^2+y^2+z^2 \leq 4$ in the first octant.

Solution: Going to cylindrical coordinates, $x = \rho \sin \phi \cos \theta$, $y = \rho \sin \phi \sin \theta$, $z = \rho \cos \phi$. Since the domain is the ball of radius 4, we see that the limits of ρ are from 0 to 2. Again since it is cut by the xy -plane below, ϕ varies from 0 to $\pi/2$. The projection is the circle in the first quadrant with radius 2. So θ varies from 0 to $\pi/2$. Hence,

$$\begin{aligned} I &= \int_{\rho=0}^2 \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{\pi/2} \rho \sin \phi \cos \theta (\rho^2 \sin \phi) \, d\rho \, d\theta \, d\phi \\ &= 4 \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{\pi/2} \sin^2 \phi \cos \theta \, d\phi \, d\theta = \pi \end{aligned}$$

Volume of solids of revolution: Consider a function $f(x) \geq 0$, $a \leq x \leq b$ and revolve it around x -axis. Then the region can be represented as

$$R = \{(x, y, z) : a \leq x \leq b, \sqrt{y^2+z^2} \leq f(x)\}$$

Taking the cylindrical coordinates $x = x, y = r \sin \theta, z = r \cos \theta$, we get

$$R = \{(x, r, \theta) : a \leq x \leq b, 0 \leq r \leq f(x), 0 \leq \theta \leq 2\pi\}$$

Therefore, the voloume is

$$\begin{aligned} V &= \int_a^b \int_0^{2\pi} \int_0^{f(x)} r dr d\theta dx \\ &= 2\pi \int_a^b \frac{[f(x)]^2}{2} = \pi \int_a^b (f(x))^2 dx \end{aligned}$$

3 Surface Area and Surface integrals

3.1 Surface Area

Consider a surface S defined with $f(x, y, z) = c$. Let R be its projection on xy -plane. Assume that this projection is **one-one, onto**. Let R_k be a small rectangle with area ΔA_k and let $\Delta \sigma_k$ be the piece of surface above this rectangle. Let ΔP_k be the tangent plane at (x_k, y_k, z_k) of the surface $\Delta \sigma_k$. Now consider the parallelogram with ΔP_k and ΔA_k as upper and lower planes of the parallelogram. We approximate the area of the surface with the area of the tangent plane ΔP_k .

Now let $\hat{p}$ be the unit normal to the plane containing R_k and ∇f is the normal to the surface. Let u_k, v_k be the vectors along the sides of the tangent plane ΔP_k . Then the area of ΔP_k is $|u_k \times v_k|$ and $u_k \times v_k$ is the normal vector to ΔP_k . Thus ∇f and $u_k \times v_k$ are both normals to the tangent plane ΔP_k .

The angle between the plane ΔA_k and ΔP_k is same as the angle between their normals. i.e., the angle between $\hat{p}$ and $u_k \times v_k$. From the geometry, the area of the projection of this tangent plane is $|(u_k \times v_k) \cdot \hat{p}|$ (proof of this can be seen in Thomas calculus Appendix 8). i.e.,

$$\Delta A_k = |(u_k \times v_k) \cdot \hat{p}| = |(u_k \times v_k)| |\hat{p}| \cos(\text{angle between } (u_k \times v_k) \text{ and } \hat{p})$$

In other words,

$$|\Delta P_k| \cos \gamma_k = \Delta A_k \text{ or } \Delta P_k = \frac{\Delta A_k}{|\cos \gamma_k|}$$

where γ_k = angle between $(u_k \times v_k)$ and $\hat{p}$. This angle can be calculated easily by noting that ∇f and $u_k \times v_k$ are both normals to the tangent plane.

(This formula is simple in case of straight lines: Let OP be the line from origin and let R be the projection of P on x -axis. Then $OR = OP \cos \gamma$ where γ is the angle between OP and OR . Now imagine the Area of plane is nothing but "sum" of lengths of lines.)

So

$$|\nabla f \cdot \hat{p}| = |\nabla f| |\hat{p}| \cos \gamma_k$$

Therefore,

$$\text{Surface Area} \approx \sum_k \Delta P_k = \sum_k \frac{|\nabla f|}{|\nabla f \cdot \hat{p}|} \Delta A_k$$

This sum converges to

$$\text{Surface Area} = \iint_R \frac{|\nabla f|}{|\nabla f \cdot \hat{p}|} dA$$

where R is the projection of S on to one of the planes and $\hat{p}$ is the unit normal to the plane of projection.

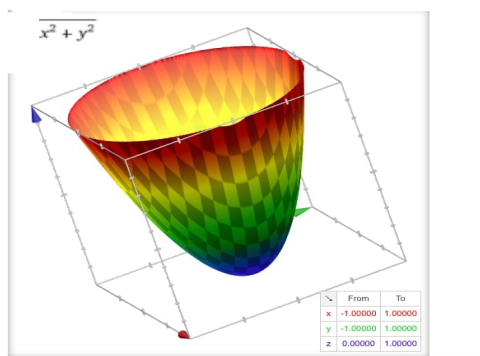


Figure 3: Paraboloid cut by $z=2$.

Example 3.1.1 Find the surface area of the curved surface of paraboloid $z = x^2 + y^2$ that is cut by the plane $z = 2$.

Solution: The equation of surface is $f(x, y, z) = z - x^2 - y^2 = 0$. Clearly this is one-one from xy -plane to $\mathbb{R}^3$. So the projection of the surface $\{(x, y, x^2 + y^2) : x^2 + y^2 \leq 2\}$ is the disc $R : x^2 + y^2 \leq 2$. Since the plane of projection is xy -plane, $\hat{p} = \hat{k}$. Hence

$$\nabla f = -2x\hat{i} - 2y\hat{j} + \hat{k}$$

$$\begin{aligned} S &= \iint_R \frac{|\nabla f|}{|\nabla f \cdot \hat{p}|} dA \\ &= \iint_R \sqrt{4x^2 + 4y^2 + 1} dA \end{aligned}$$

Going to polar coordinates $x = r \cos \theta, y = r \sin \theta$,

$$S = \int_0^{2\pi} \int_{r=0}^{\sqrt{2}} \sqrt{1 + 4r^2} r dr d\theta = 13\pi$$

Example 3.1.2 Find the surface area of the cap obtained by cutting the hemisphere $x^2 + y^2 + z^2 = 2$ by the cone $z = \sqrt{x^2 + y^2}$.

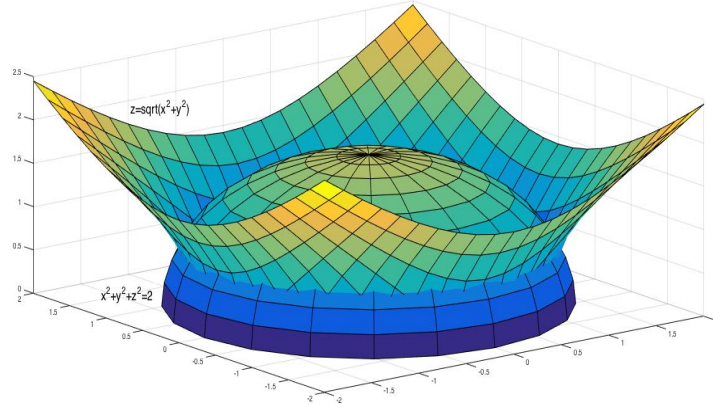


Figure 4: Cap obtained by cutting the hemisphere by the cone.

Solution: The equation of surface is $f(x, y, z) = x^2 + y^2 + z^2 - 2 = 0$ and we can take the projection onto xy -plane. So $\hat{p} = \hat{k}$. The projection is obtained by solving $x^2 + y^2 + z^2 = 2, z = \sqrt{x^2 + y^2}$. i.e., $R = x^2 + y^2 = 1$.

$$\nabla f = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

$$|\nabla f \cdot \hat{p}| = 2z = 2\sqrt{2 - x^2 - y^2}$$

Therefore, using polar coordinates $x = r \cos \theta, y = r \sin \theta$,

$$\begin{aligned} S &= \iint_R \frac{\sqrt{2}}{\sqrt{2 - x^2 - y^2}} dA \\ &= \int_0^{2\pi} \int_0^1 \frac{\sqrt{2}}{\sqrt{2 - r^2}} r \, dr \, d\theta = 2\pi(2 - \sqrt{2}). \end{aligned}$$

Surface area of solids of revolution: If a surface is generated by revolving a curve $z = f(y), y \in (0, \rho)$ in the yz -plane about the z -axis. This surface is the graph of function $z = f(\sqrt{x^2 + y^2})$. Then by taking $g(x, y, z) = z - f(\sqrt{x^2 + y^2})$ over the circular domain R which is the projection of the solid. So, we get

$$S = \iint_R \frac{|\nabla g|}{|\nabla g \cdot \hat{k}|} dxdy = \iint_R (1 + f'(\sqrt{x^2 + y^2})) dxdy$$

Now using polar coordinates $x = r \cos \theta, y = r \sin \theta$, we get

$$S = \int_0^{2\pi} \int_0^\rho \sqrt{1 + (f'(r))^2} r \, dr \, d\theta = 2\pi \int_{r=0}^\rho \sqrt{1 + (f'(r))^2} r \, dr$$

3.2 Surface Integrals

Let $g(x, y, z)$ be a function defined over a surface S . Then we can think of integration of g over S . Suppose, a surface S is heated up, we have a temperature distributed over this surface. Let $T(x, y, z)$ be the temperature at (x, y, z) of the surface. Then we can calculate the total temperature on S using the Riemann integration.

Let R be the projection of S on the plane. We partition R into small rectangles A_k . Let ΔS_k be the surface above the ΔA_k . We approximate this surface area element with its tangent plane ΔP_k . As we refine the rectangular partition this ΔP_k approximated the ΔS_k . Then the total temperature may be approximated as

$$\sum_k g(x_k, y_k, z_k) \Delta P_k = \sum_k g(x_k, y_k, z_k) \frac{\Delta A_k}{|\cos \gamma_k|} = \sum_k g(x_k, y_k, z_k) \frac{|\nabla f|}{|\nabla f \cdot \hat{p}|} dA$$

where $\hat{p}$ is the unit normal to R or the plane of projection. Now taking limit $n \rightarrow \infty$, we get

$$\iint_S g(x, y, z) dS = \iint_R g(x, y, z) \frac{|\nabla f|}{|\nabla f \cdot \hat{p}|} dA$$

If the surface is defined as $f = z - h(x, y) = 0$, then

$$\iint_S g(x, y, z) dS = \iint_R g(x, y, h(x, y)) \frac{|\nabla f|}{|\nabla f \cdot \hat{k}|} dA.$$

Example 3.2.1 Integrate $g(x, y, z) = z$ over the surface S cut from the cylinder $y^2 + z^2 = 1, z \geq 0$, by the planes $x = 0$ and $x = 1$.

Solution: $f = y^2 + z^2$ and this surface can be projected onto to R of xy plane. This projection is the rectangle with vertices $(1, -1), (1, 1), (0, 1), (0, -1)$. So $\hat{p} = \hat{k}$

$$\frac{|\nabla f|}{|\nabla f \cdot \hat{p}|} = \frac{2\sqrt{y^2 + z^2}}{|2z|} = \frac{1}{z}$$

Therefore,

$$\iint_S z dS = \iint_R z \frac{1}{z} dA = \text{Area}(R) = 2$$

Parametrizations of Surfaces: Let

$$\mathbf{r}(u, v) = f(u, v)\hat{i} + g(u, v)\hat{j} + h(u, v)\hat{k}$$

be a continuous vector function defined on a plane region R . The variable u and v are parameters and R is the parameter domain. The range of r is called the surface S . We assume that r is one-to-one on the interior of R so that S does not cross itself.

Examples 3.2.2 1. A parametrization of the cone

$$z = \sqrt{x^2 + y^2}, \quad 0 \leq z \leq 1$$

Here cylindrical coordinates provide everything we need. Let

$$x(r, \theta) = r \cos \theta, \quad y(r, \theta) = r \sin \theta, \quad z = \sqrt{x^2 + y^2} = r$$

the domain of r, θ is $0 \leq r \leq 1$ and $0 \leq \theta \leq 2\pi$. So the required parametrization is

$$\mathbf{r}(r, \theta) = r \cos \theta \hat{i} + r \sin \theta \hat{j} + r \hat{k}, \quad 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi.$$

2. A parametrization of sphere

$$x^2 + y^2 + z^2 = a^2$$

Here we take spherical coordinates. Let $x = a \sin \phi \cos \theta, y = a \sin \phi \sin \theta$, and $z = a \cos \phi$, $0 \leq \phi \leq \pi, 0 \leq \theta \leq 2\pi$. Taking $u = \phi$ and $v = \theta$, we get

$$\mathbf{r}(\theta, \phi) = a \sin \phi \cos \theta \hat{i} + a \sin \phi \sin \theta \hat{j} + a \cos \phi \hat{k}, \quad 0 \leq \phi \leq \pi, 0 \leq \theta \leq 2\pi.$$

3. A parametrization of cylinder $x^2 + (y - 3)^2 = 9, 0 \leq z \leq 5$
we take cylindrical coordinates,

$$x(r, \theta) = r \cos \theta, \quad y(r, \theta) = r \sin \theta, \quad z = z$$

Substituting in $x^2 + (y - 3)^2 = 9$ we get $r^2 + 6r \cos \theta = 0$. Therefore, $r = 6 \sin \theta$.

$$\mathbf{r}(\theta, z) = 3 \sin 2\theta \hat{i} + 6 \sin^2 \theta \hat{j} + z \hat{k}, \quad 0 \leq \theta \leq \pi, 0 \leq z \leq 5$$

Definition 3.2.3 (Smooth surface): A parametrized surface $\mathbf{r}(u, v)$ is called smooth surface if $\mathbf{r}_u$ and $\mathbf{r}_v$ are continuous and $\mathbf{r}_u \times \mathbf{r}_v$ is never zero.

Surface Area: We approximate the surface area element by the parallelogram on the tangent plane whose sides are determined by the vectors $r_u \Delta u$ and $r_v \Delta v$. The total surface area is approximately equal to the sum of area of of this parallelograms

$$S \sim \sum_u \sum_v |r_u \times r_v| \Delta u \Delta v$$

This sum is a Riemann sum of the integral $\int_a^b \int_a^b |r_u \times r_v| \, du \, dv$. Therefore we have

The Surface area of smooth surface: $\mathbf{r}(u, v) = f(u, v)\hat{i} + g(u, v)\hat{j} + h(u, v)\hat{k}$, $a \leq u \leq b$, $c \leq v \leq d$ is

$$S = \int_a^b \int_c^d |\mathbf{r}_u \times \mathbf{r}_v| \, du \, dv$$

Problem 3.2.1 Find the surface area of the surface of the cone

$$z = \sqrt{x^2 + y^2}, \quad 0 \leq z \leq 1$$

Solution: We found a parametrization of the cone as

$$\mathbf{r}(r, \theta) = r \cos \theta \hat{i} + r \sin \theta \hat{j} + r \hat{k}, \quad 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi.$$

We can find that

$$|\mathbf{r}_r \times \mathbf{r}_\theta| = \sqrt{r^2 \cos^2 \theta + r^2 \sin^2 \theta + r^2} = \sqrt{2}r$$

Therefore,

$$\text{Surface Area} = \int_0^{2\pi} \int_0^1 \sqrt{2}r dr d\theta = \pi\sqrt{2}$$

Surface integrals: Let $F(u, v)$ be a continuous function defined on the parametrized surface S : $\mathbf{r}(u, v) : R \rightarrow S$, where $R : a \leq u \leq b, c \leq v \leq d$. Then

$$\iint_S F dS = \int_a^b \int_c^d F(u, v) |\mathbf{r}_u \times \mathbf{r}_v| du dv$$

Problem 3.2.2 Evaluate the surface integral $\iint_S (x + y + z) dS$ over the surface of cylinder $x^2 + y^2 = 9, 0 \leq z \leq 4$.

Solution: Using the cylindrical coordinates: $x = 3 \cos \theta, y = 3 \sin \theta, z = z$ over the parameter domain $\{(\theta, z) : 0 \leq \theta \leq 2\pi, 0 \leq z \leq 4\}$. The surface can be represented as

$$T(\theta, z) = 3 \cos \theta \hat{i} + 3 \sin \theta \hat{j} + z \hat{k}.$$

Then $|T_\theta \times T_z| = \sqrt{9 \cos^2 \theta + 9 \sin^2 \theta} = 3$. The given integral is equal to

$$\begin{aligned} \iint_S (x + y + z) dS &= \iint_S (3 \cos \theta + 3 \sin \theta + z) |T_\theta \times T_z| d\theta dz \\ &= \int_{z=0}^4 \int_{\theta=0}^{2\pi} (3 \cos \theta + 3 \sin \theta + z) d\theta dz = 6\pi \int_0^4 z dz = 48\pi. \end{aligned}$$

4 Line integrals and Green's theorem

In many physical phenomena, the integrals over paths through vector fields play an important role. For example, work done in moving an object along a path against a variable force or to find work done by a vector field in moving an object along a path through the field. A **vector field** on a domain in the plane or in the space is a vector valued function $f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ with components say M, N and P , for example

$$F(x, y, z) = M(x, y, z) \hat{i} + N(x, y, z) \hat{j} + P(x, y, z) \hat{k}$$

We assume that M, N, P are continuous functions. Suppose F represents a force throughout a region in space and let $r(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$, $a \leq t \leq b$ is a smooth curve in the region. Then we introduce the partition $a = t_1 < t_2 \dots < t_n = b$ of $[a, b]$.

If F_k denotes the value of F at the point on the curve corresponding to t_k on the curve and T_k denotes the curve's unit tangent vector at this point. Then $F_k \cdot T_k$ is the scalar component of F in the direction of T at t_k . Then the work done by F along the curve is approximately

$$\sum_{k=1}^n F_k \cdot T_k \Delta s_k,$$

where Δs_k is the length of the curve between t_{k-1}, t_k . As the norm of the partition approaches zero, this sum approaches

$$\int_{t=a}^b F \cdot T ds = \int_a^b \vec{F} \cdot \vec{T} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt.$$

Now substituting $T(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$, we get

$$\int_a^b \vec{F} \cdot \vec{r}'(t) dt.$$

The following can be shown:

Theorem 4.0.1 *The Line integral is independent of choice of parametrization.*

Definition 4.0.1 *The **orientation** of a parameterized curve is the direction determined by increasing values of the parameter.*

The line integral over a parametrized curve depends on the orientation. If we change the orientation, then the integral is equal to -1 times: That is,

$$-\int_C F \cdot dr = \int_{-C} F \cdot dr.$$

Example 4.0.2 *Find the work done by $F = 3x^2\hat{i} + (2xz - y)\hat{j} - z\hat{k}$ over the curve $r(t) = t\hat{i} + t^2\hat{j} + t^3\hat{k}$, $0 \leq t \leq 1$ from origin to $(1, 1, 1)$*

Solution: The tangent along the curve T is $\frac{dr}{dt}$. Therefore,

$$\begin{aligned} \int_0^1 F \cdot T ds &= \int_0^1 F \cdot \frac{d\vec{r}}{dt} dt \\ &= \int_0^1 (3t^2 + t^5 - 2t^3) dt = \frac{2}{3}. \end{aligned}$$

Definition 4.0.3 *Conservative vector field: A vector field $\vec{F}$ is called conservative vector field if the line integral depends only on the end points. Equivalently, the line integral over any closed curve is zero.*

The following is known as fundamental theorem of Line integrals. Recall the fundamental theorem of integral calculus.

Theorem 4.0.2 Let $\vec{F}$ be a vector field and if there exists a differentiable function $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ such that $\vec{F} = \nabla f$. Then

$$\int_P^Q \vec{F} \cdot d\vec{r} = f(Q) - f(P)$$

Proof. Suppose there exists f such that $\vec{F} = \nabla f$. let $\vec{r}(t) : \{(x(t), y(t), z(t)), t \in [0, 1]\}$ represent a curve connecting P and Q . Then

$$\begin{aligned} \int_P^Q \vec{F} \cdot d\vec{r} &= \int_0^1 \left(\frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} + \frac{\partial f}{\partial z} \frac{dz}{dt} \right) dt \\ &= \int_0^1 \frac{d}{dt} (f(x(t), y(t), z(t))) dt \\ &= f(Q) - f(P). \end{aligned}$$

It is clear that if $P = Q$, then above integral is zero.

Theorem 4.0.3 Let $\vec{F} = M\hat{i} + N\hat{j} + P\hat{k}$ be a conservative whose components are continuous over an open connected domain D in $\mathbb{R}^3$. Then there exists a differential function f such that $\vec{F} = \nabla f$.

Proof. let X_0 be a fixed point in D . For any point (x, y, z) in D , let C be a path from X_0 to (x, y, z) . Define $f(x, y, z)$ by

$$f(x, y, z) = \int_C \vec{F} \cdot d\vec{r}.$$

(Note that this definition of f makes sense only because it is independent of path). We want to show that f satisfies $\nabla f = \vec{F}$.

Since domain D is open, it is possible to find a disk centered at (x, y, z) such that the disk is contained entirely inside D . Let (a, y, z) with $a < x$ be a point in that disk. Let C be a path from X_0 to (x, y, z) that consists of two pieces: C_1 and C_2 . The first piece, C_1 , is any path from C to (a, y, z) that stays inside D and C_2 is the horizontal line segment from (a, y, z) to (x, y, z) . Then

$$f(x, y, z) = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r}$$

Then differentiating this with respect to x , we see that the $\frac{\partial}{\partial x}$ of the first term on the right hand side is zero. Therefore

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x} \int_{C_2} \vec{F} \cdot d\vec{r}$$

Now considering the parametrization $\vec{r}(t) = t\hat{i} + y\hat{j} + z\hat{k}$, $a \leq t \leq x$. Then $\vec{F} \cdot d\vec{r} = M(t, y, z) dt$. Hence

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x} \int_a^x M(t, y, z) dt = M(x, y, z)$$

thanks to fundamental theorem of integral calculus. A similar argument using a lines parallel to y -axis and z -axis rather than a line parallel to x -axis, shows that $f_y = N(x, y, z)$ and $f_z = N(x, y, z)$. $\square$

The following is a necessary condition.

Theorem 4.0.4 Suppose F is a conservative vector field with all its components are differentiable and partial derivatives are continuous in the domain D . Then $\nabla \times \vec{F} = 0$.

Proof. Since $\vec{F}$ is conservative, there exists f such that $\vec{F} = \nabla f$. Therefore, second order partial derivatives are continuous. Hence mixed derivatives are equal. Now it is easy to check that $\nabla \times \nabla f = 0$. Therefore $\nabla \times \vec{F} = 0$. $\square$

Definition 4.0.4 A vector field is irrotational if $\nabla \times \vec{F} = 0$.

Now we ask the question: Is irrotational vector field is always conservative?

The answer is NO. The following example

Example 4.0.5 Let $\vec{F} = F_1\hat{i} + F_2\hat{j} = -\frac{y}{x^2+y^2}\hat{i} + \frac{x}{x^2+y^2}\hat{j}$. Then $\frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$ implying $\nabla \times \vec{F} = 0$. But the line integral $\int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \sin^2 \theta + \cos^2 \theta = 2\pi$.

But depending on the shape of the domain D , $\nabla \times \vec{F} = 0$ implies $\vec{F}$ is conservative.

Definition 4.0.6 A subset D of $\mathbb{R}^n$ is called simply connected if it is path-connected and every loop in D can be contracted to a point without ever leaving the domain.

Examples 4.0.7 1. The whole space $\mathbb{R}^n$ is simply connected.

2. The unit ball $\{(x, y, z) : x^2 + y^2 + z^2 < 1\}$ is simply connected.

3. The annulus $\{(x, y, z) : 1 < x^2 + y^2 + z^2 < 2\}$ is NOT connected.

4. The punctured disc $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\} \setminus \{(0, 0)\}$ is NOT simply connected.

5. The punctured space $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 < 1\} \setminus \{(0, 0, 0)\}$ is simply connected.

6. $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 < 1\} \setminus \{z\text{-axis}\}$ is NOT simply connected.

Theorem 4.0.5 Let $\vec{F}$ be a vector field with its components are differentiable and partial derivatives are continuous. Then $\nabla \times \vec{F} = 0$ implies $\vec{F}$ is conservative.

Proof. Proof requires Stoke's theorem. We omit it.

Example 4.0.8 Consider the vector field $\vec{F} = (3x^2y^2z + 5y^3)\hat{i} + (2x^3yz + 15xy^2 - 7z)\hat{j} + (x^3y^2 - 7y + 4z^3)\hat{k}$ Determine whether $\vec{F}$ is conservative, and if it is, find a potential function f for which $\vec{F} = \nabla f$.

Solution It is easy to check that $\nabla \times \vec{F} = 0$ and domain of definition of $\vec{F}$ is the whole space $\mathbb{R}^3$ which is simply connected. Therefore, $\vec{F}$ is conservative.

To find the function f , we take

$$f_x = 3x^2y^2z + 5y^3, \quad f_y = 2x^3yz + 15xy^2 - 7z, \quad f_z = x^3y^2 - 7y + 4z^3$$

Using the first equation we obtain

$$f(x, y, z) = \int (3x^2y^2z + 5y^3)dx = x^3y^2z + 5y^3x + g(y, z)$$

Differentiation with respect to y and equating with second equation we get

$$2x^3yz + 15y^2x + g_y(y, z) = 2x^3yz + 15xy^2 - 7z \implies g_y = -7z$$

Integrating g_y we get $g(y, z) = -7yz + h(z)$. Hence $f = x^3y^2z + 5y^3x - 7yz + h(z)$. Substituting this the third equation, we get $h'(z) = 4z^3$. Therefore $h(z) = z^4 + c$. Hence $f(x, y, z) = x^3y^2z + 5y^3x - 7yz + z^4 + c$. $\square$

Example 4.0.9 Consider the vector field $\vec{F} = \frac{1}{x^2+y^2+z^2}(x\hat{i} + y\hat{j} + z\hat{k})$. Determine whether $\vec{F}$ is conservative, and if it is, find a potential function f for which $\vec{F} = \nabla f$.

Solution: In this case again the domain of definition is the punctured space $\mathbb{R}^3 \setminus \{(0, 0, 0)\}$ which is simply connected. Also it is easy to check

$$\nabla \times \vec{F} = \frac{-2}{(x^2 + y^2 + z^2)^2} \left((zy - yz)\hat{i} + (xz - zx)\hat{j} + (yx - xy)\hat{k} \right) = 0.$$

Now following the steps as above, it is not difficult to find f as

$$f(x, y, z) = \frac{1}{2} \ln(x^2 + y^2 + z^2).$$

Theorem 4.0.6 Green's theorem in the plane

Let R be a closed bounded region in $\mathbb{R}^2$ whose boundary C consists of finitely many smooth curves. Let $\vec{F}(x, y) = F_1(x, y)\hat{i} + F_2(x, y)\hat{j}$ be continuous and has continuous partial derivatives everywhere in some domain containing R . Then

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA = \oint_C \vec{F} \cdot d\vec{r},$$

where the line integral is along the boundary C of R such that R is on the left as we advance on the boundary.

Example 4.0.10 Evaluate $\int_C \vec{F} \cdot d\vec{r}$ for $\vec{F} = (y^2 - 7y)\hat{i} + (2xy + 2x)\hat{j}$ and $C : x^2 + y^2 = 1$.

Solution:

$$\int_C \vec{F} \cdot d\vec{r} = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA = 9 \iint_R dA = 9\pi.$$

Remark 4.0.1 The region R is such that the boundary consists of smooth curves. The theorem does not hold true for regions like punctured disc $\{(x, y) : x^2 + y^2 \leq 1\} \setminus \{0\}$. For example, take $\vec{F} = -\frac{y}{x^2 + y^2}\hat{i} + \frac{x}{x^2 + y^2}\hat{j}$. Then $\frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$ and $\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) = 0$. But the line integral $\int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \sin^2 \theta + \cos^2 \theta = 2\pi$.

Area of plane region:

Using Green's theorem, we can write area of a plane region as a line integral over the boundary. Choose $F_1 = 0, F_2 = x$ and then $F_1 = -y, F_2 = 0$. This gives

$$\iint_R dA = \int_C x dy \quad \text{and} \quad \iint_R dA = - \int_C y dx$$

respectively. The double integral is the area A of R . By addition we have

$$A = \frac{1}{2}(x dy - y dx)$$

Example 4.0.11 Area bounded by ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

solution: Take $x = a \cos t, y = b \sin t, 0 \leq t \leq 2\pi$. Then by above formula

$$A = \frac{1}{2} \int_0^{2\pi} (xy' - yx') dt = \frac{1}{2} (ab \cos^2 t - (-ab \sin^2 t)) dt = \pi ab$$

5 Gauss and Stokes theorems

Let S be a smooth surface and we may choose unit normal $\hat{n}$ at P of S . The direction of $\hat{n}$ is called positive normal direction of S at P . We call a smooth surface S orientable surface if the positive normal at P can be continued in a unique and continuous way to the entire surface. For example the Mobius strip is not orientable. A normal at a point P of this strip is displaced continuously along a closed curve C , the resulting normal upon returning to P is opposite to the original vector at P .

Theorem 5.0.1 Gauss Divergence Theorem

Let Ω be a closed, bounded region in $\mathbb{R}^3$ whose boundary is a piecewise smooth orientable surface S . Let $\vec{F}(x, y, z)$ be a continuous function that has continuous partial derivatives in some domain containing Ω . Then

$$\iiint_{\Omega} \nabla \cdot F dV = \iint_S \vec{F} \cdot \hat{n} dS$$

where $\hat{n}$ is the outer unit normal vector of S .

Example 5.0.1 Evaluate $\iint_{\partial\Omega} \vec{F} \cdot \hat{n} dA$ where $\partial\Omega$ is the boundary of the domain inside the cylinder $x^2 + y^2 = 1$ and between the planes $z = 0, z = x + 2$ and $\vec{F} = (x^2 + ye^z)\hat{i} + (y^2 + ze^2)\hat{j} + (z^2 + xe^y)\hat{k}$.

Solution: With the given $\vec{F}$, it is not difficult to obtain, $\nabla \cdot \vec{F} = 2x + 2y + 2z$. By Divergence theorem

$$\iint_{\partial\Omega} \vec{F} \cdot \hat{n} dS = \iiint_{\Omega} 2(x + y + z) dV = 2 \iint_{x^2+y^2 \leq 1} \left(\int_{z=0}^{x+2} (x + y + z) dz \right) dx dy$$

Theorem 5.0.2 Stokes's theorem

Let S be a piecewise smooth oriented surface with boundary and let boundary C be a simple closed curve. Let $\vec{F}$ be a continuous function which has continuous partial derivatives in a domain containing S . Then

$$\iint_S (\nabla \times \vec{F}) \cdot \hat{n} dS = \int_C \vec{F} \cdot d\vec{r}$$

where $\hat{n}$ is a unit normal vector of S and, depending on $\hat{n}$, the integration around C is taken in the way that S lies in the left of C . Here $\hat{n}$ is the direction of your head while moving along the boundary with surface on your left.

Example 5.0.2 Evaluate $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = x^2 y^3 \hat{i} + \hat{j} + z \hat{k}$ and C The intersection of the cylinder $x^2 + y^2 = 4$ and the hemisphere $x^2 + y^2 + z^2 = 16, z \geq 0$.

Solution: The intersection of cylinder and sphere is the boundary of cylinder on the plane $z = \sqrt{12}$. The unit normal to the surface is $\hat{n} = \frac{1}{4}(x\hat{i} + y\hat{j} + z\hat{k})$. The projection R of S on the xy -plane is the disc $x^2 + y^2 \leq 2$, $\nabla \times \vec{F} = -3x^2 y^2 \hat{k}$ and $\frac{|\nabla f|}{|\nabla f \cdot \hat{p}|} = \frac{4}{z}$. Hence by Stoke's theorem

$$\begin{aligned} \oint_C \vec{F} \cdot d\vec{r} &= \iint_R \left(-\frac{3}{4}\right) x^2 y^2 z \frac{4}{z} dA \\ &= -3 \int_{\theta=0}^{2\pi} \int_{r=0}^2 (r^2 \cos^2 \theta)(r^2 \sin^2 \theta) r dr d\theta = -8\pi. \end{aligned}$$

Suppose S_1, S_2 be two surfaces having the same boundary curve C . An important consequence of Stoke's theorem is that flux through S_1 or S_2 is same.

Example 5.0.3 Suppose S is a surface of a light bulb over the unit disc $x^2 + y^2 = 1$ oriented with outward pointing normal. Suppose $\vec{F} = e^{z^2-2z} x \hat{i} + (\sin(xyz) + y + 1) \hat{j} + e^{z^2} \sin(z^2) \hat{k}$. Compute $\iint_S (\nabla \times \vec{F}) \cdot \hat{n} dS$.

Solution: Enough to take any surface with boundary $x^2 + y^2 = 1$. So we take the unit disc $x^2 + y^2 \leq 1, z = 0$. Then $\vec{F}$ on this is $\vec{F} = x \hat{i} + (y+1) \hat{j}$. Then $\nabla \times \vec{F} = 0$. Hence $\int_C \vec{F} \cdot d\vec{r} = 0$.

References

1. Thomas' Calculus, Chapters 15, 16
2. Advanced engineering Mathematics, E.Kreyszig, Chapter 9

5.1 Problems

1. Evaluate the double integrals:

(a) $\iint_R x^2 dA$, where R is the region bounded by $y = x^2, y = x + 2$

(b) $\iint_R (x^2 + y^2) dA$, where $R : 0 \leq y \leq \sqrt{1 - x^2}, 0 \leq x \leq 1$.

(c) $\iint_R (a^2 - x^2 - y^2) dA$, where R is the region $x^2 + y^2 \leq a^2$.

2. Evaluate the following double integrals by changing the order of integration if needed:

(a) $\int_0^3 \int_{-y}^y (x^2 + y^2) dx dy$ (b) $\int_0^\pi \int_0^\pi |\cos(x + y)| dx dy$

(c) $\int_0^1 \int_2^{4-2x} dy dx$ (d) $\int_0^{3/2} \int_0^{9-4x^2} 16x dy dx$

(e) $\int_0^\pi \int_x^\pi \frac{\sin y}{y} dy dx$ (f) $\int_0^2 \int_0^{4-x^2} \frac{xe^{2y}}{4-y} dy dx$

3. Find the volume of the following:

(a) Region under the paraboloid $z = x^2 + y^2$ and above the triangle enclosed by the lines $y = x, x = 0$, and $x + y = 2$ in the xy plane.

(b) Region bounded above by the cylinder $z = x^2$ and below by the region enclosed by the parabola $y = 2 - x^2$ and the line $y = x$ in the xy plane.

(c) Region bounded in the first octant bounded by the coordinate planes, the cylinder $x^2 + y^2 = 4$, and the plane $z + y = 3$.

(d) Solid cut from the first octant by the cylinder $z = 12 - 3y^2$ and the plane $x + y = 2$.

(e) Tetrahedron bounded by the planes $y = 0, z = 0, x = 0$ and $-x + y + z = 1$.

4. Use the given transformations to transform the integrals and evaluate them:

(a) $u = 3x + 2y, v = x + 4y$ and $I = \iint_R (3x^2 + 14xy + 8y^2) dA$ where R is the region in the first quadrant bounded by the lines $y + \frac{3}{2}x = 1, y + \frac{3}{2}x = 3, y + \frac{1}{4}x = 0$, and $y + \frac{1}{4}x = 1$.

(b) $u = x + 2y, v = x - y$ and $I = \int_0^{2/3} \int_y^{2-2y} (x + 2y)e^{(y-x)} dA$

(c) $u = xy, v = x^2 - y^2$ and $I = \iint_R (x^2 + y^2) dA$, where R is the region bounded by $xy = 1, xy = 2, x^2 - y^2 = 1$ and $x^2 - y^2 = 2$.

5. Using appropriate transformation evaluate $\iint_R dA$, where R is the parallelogram with vertices $(1, 0), (3, 1), (2, 2)$ and $(0, 1)$.

6. Find the area of the following:

- (a) The region lies inside the cardioid $r = 1 + \cos \theta$ and outside the circle $r = 1$ in the first quadrant.
- (b) The region common to the interiors of the cardioids $r = 1 + \cos \theta$ and $r = 1 - \cos \theta$.

7. Find the volume of the following solids:

- (a) Cylinder whose base lies inside the cardioid $r = 1 + \cos \theta$ and outside the circle $r = 1$ and the top lies in the plane $z = x$.
- (b) Cylinder whose base is enclosed by $r^2 = 2 \cos 2\theta$ and top is bounded by the sphere $z = \sqrt{2 - r^2}$.

8. Evaluate the following volume integrals:

- (a) $\iiint_D (z^2 x^2 + z^2 y^2) dV$, where $D = \{(x, y, z) \in \mathbb{R}^3, x^2 + y^2 \leq 1, -1 \leq z \leq 1\}$
- (b) $\iiint_D xyz dV$ where $D = \{(x, y, z) \in \mathbb{R}^3, x^2 + y^2 \leq 1, 0 \leq z \leq x^2 + y^2\}$
- (c) $\iiint_D e^{(x^2+y^2+z^2)^{3/2}} dV$ where $D = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1\}$

9. Find the volume of the following regions using triple integrals:

- (a) The region in the first octant bounded by the coordinate planes and the planes $x + z = 1, y + 2z = 2$.
- (b) The region in the first octant bounded by the coordinate planes, the plane $y + z = 2$, and the cylinder $x = 4 - y^2$.
- (c) The tetrahedron in the first octant bounded by the coordinate planes and the plane $x + y/2 + z/3 = 1$.
- (d) The region common to the interiors of the cylinders $x^2 + y^2 = 1$ and $x^2 + z^2 = 1$.
- (e) The region cut from the cylinder $x^2 + y^2 = 4$ by the plane $z = 0$ and the plane $x + z = 3$.
- (f) The region enclosed by $y = x^2, y = x + 2, 4z = x^2 + y^2$ and $z = x + 3$.
- (g) The region bounded above by the sphere $x^2 + y^2 + z^2 = 2$ and below by the paraboloid $z = x^2 + y^2$.
- (h) The solid bounded by the cone $z = \sqrt{x^2 + y^2}$ and paraboloid $z = x^2 + y^2$.

10. Find the mass of the solid, center of mass, moment of inertia and the radii of gyration of a solid cube in the first octant bounded by $x = 1, y = 1$ and $z = 1$. The density $\delta(x, y, z) = x + y + z + 1$

11. Find the mass of solid and center of mass of a solid in the first octant bounded by the planes $y = 0, z = 0$ and by the surfaces $z = 4 - x^2$ and $x = y^2$ and its density is $\delta(x, y, z) = kxy, k$ is a constant.
12. Using the given transformation transform the given integrals and evaluate them:
 - (a) $x = au, y = bv, z = cw$, and $I = \iiint_D dV$ where D is the ellipsoid: $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.
 - (b) $u = x, v = xy, w = 3z$ and $I = \iiint_D (x^2y + 3xyz) dV$ where $D = \{(x, y, z) \in \mathbb{R}^3 : 1 \leq x \leq 2, 0 \leq xy \leq 2, 0 \leq z \leq 1\}$.
13. Find the surface area of the following:
 - (a) The surface cut from the paraboloid $x^2 + y^2 - z = 0$ by the plane $z = 2$.
 - (b) The surface of the region cut from the plane $x + 2y + 2z = 5$ by the cylinder whose walls are $x = y^2$ and $x = 2 - y^2$.
 - (c) The cap cut from the sphere $x^2 + y^2 + z^2 = 2$ by the cone $z = \sqrt{x^2 + y^2}$.
 - (d) The surface cut from the paraboloid $x^2 + y + z^2 = 2$ by the plane $y = 0$.
 - (e) The surface cut from the cylinder $x^2 + y^2 = 4$ by the cylinder $y^2 + z^2 = 4$.
 - (f) The surface cut from the cone $x^2 + y^2 = z^2$ by $x^2 + y^2 = 4x$.
 - (g) The surface of the cylinder $x^2 + y^2 = 4x$ which lies between the plane $z = 0$ and the cylinder $x^2 + y^2 = z^2$.
 - (h) The surface area of the part of the cylinder $x^2 + y^2 = a^2$ which lies between the plane $z = 2x$ and $z = 0$.
 - (i) The surface area of the part of the cylinder $x^2 + y^2 = 2x$ which lies above xy -plane and is bounded above by the paraboloid $x^2 + y^2 + z = 4$.
14. Evaluate $\iint_S g dS$ in each of the following cases:
 - (a) $g(x, y, z) = x + y + z$ over the surface S of the cube cut from the first octant by the planes $x = a, y = a, z = a$.
 - (b) $g(x, y, z) = y + z$ over the surface S of the wedge in the first octant bounded by the coordinate planes and the planes $x = 2$ and $y + z = 1$.
 - (c) $g(x, y, z) = xyz$ over the surface S of the rectangular solid cut from the first octant by the plane $x = a, y = b$, and $z = c$.
15. Compute the line integral $\int_C \vec{F} d\vec{r}$, where $\vec{F} = 4z\hat{i} - 3x\hat{k}$ and C is the circle $(x - 5)^2 + (z - 7)^2 = 4$ in xz plane described in anticlockwise direction.
16. Let $\vec{F} = (3x - 4y + 2z)\hat{i} + (4x + 2y - 3z^2)\hat{j} + (2xz - 4y^2 + z^3)\hat{k}$. Find the work done by $\vec{F}$ in moving a particle once around the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$.

17. Find the work done by force $\vec{F}(t) = (3y^2 + 2)\hat{i} + 16x\hat{j}$ in moving a particle from $(-1, 0)$ to $(1, 0)$ along the upper half of the ellipse $b^2x^2 + y^2 = b^2$. Find that ellipse that makes the work done minimum.
18. Evaluate the following Line integrals using Green's theorem on plane:
- (a) $\oint_C (y^2 dx + x^2 dy)$, C : The triangle bounded by $x = 0, x + y = 1, y = 0$.
 - (b) $\oint_C (3y dx + 2x dy)$, C : The boundary of $0 \leq x \leq \pi, 0 \leq y \leq \sin x$.
 - (c) $\oint_C (6y + x) dx + (y + 2x) dy$, C : The circle $(x - 2)^2 + (y - 3)^2 = 4$.
19. Using Green's theorem find the areas of regions enclosed by
- (a) The circle $\vec{r}(t) = (a \cos t)\hat{i} + (a \sin t)\hat{j}, 0 \leq t \leq 2\pi$
 - (b) The Astroid $\vec{r}(t) = (\cos^3 t)\hat{i} + (\sin^3 t)\hat{j}, 0 \leq t \leq 2\pi$
 - (c) The curve $\vec{r}(t) = t^2\hat{i} + (\frac{t^3}{3} - t)\hat{j}, -\sqrt{3} \leq t \leq \sqrt{3}$
20. Use Green's theorem to find the work done by the given force field $\vec{F}(t)$ in moving a particle once counterclockwise around the given curve C
- (a) $\vec{F}(t) = (4x - 2y)\hat{i} + (2x - 4y)\hat{j}$ and C is the circle $(x - 2)^2 + (y - 2)^2 = 4$.
 - (b) $\vec{F}(t) = x^2\hat{i} + (xy - x)\hat{j}$ and C is the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$.
21. Evaluate $\iint_S \vec{F} \cdot \hat{n} dS$, where $\hat{n}$ is the unit outward normal to S , in following cases:
- (a) $\vec{F} = yz\hat{j} + z^2\hat{k}$ and S is the curved surface cut of cylinder $y^2 + z^2 = 1, z \geq 0$ cut by the planes $x = 0$ and $x = 1$.
 - (b) $\vec{F} = xz\hat{i} + yz\hat{j} + \hat{k}$ and S is the surface of the upper cap cut from the solid sphere $x^2 + y^2 + z^2 \leq 25$ by the plane $z = 3$.
 - (c) $\vec{F} = yz\hat{i} + x\hat{j} - z^2\hat{k}$ and S is the parabolic surface of the cylinder $y = x^2, 0 \leq x \leq 1, 0 \leq z \leq 4$.
 - (d) $F = (2y, z, -x)$, and S is the part of the plane $x + y + z = 4$ that is above the rectangle in the xy -plane with vertices at $(0, 0, 0), (2, 0, 0), (0, 1, 0)$, and $(2, 1, 0)$, oriented upwards.
 - (e) $F = (ey, -ex, 0)$, and S is the portion above the xy -plane of the surface defined by $x^2 + 5y^2 - (z - 3)^2 = 16$, oriented outward (at the point $(0, 0, 7)$, the unit normal vector is $\hat{k}$).
22. Evaluate the following line integrals:
- (a) $\int_C \vec{r} \cdot d\vec{r}$, C being the helical path $x = \cos t, y = \sin t, z = t$ joining the points determined by $t = 0$ and $t = \pi/2$.

- (b) $\int_C \vec{v} \cdot d\vec{r}$, where $\vec{v} = \hat{i}y + (2x)\hat{j}$, C being an arc of a circle of radius 1 centered at the origin joining (1,0) and (0,1).
- (c) $\int_C (y-z)dx + (z+x)dy + (x-y)dz$, C being a circle formed by the intersection of the sphere $x^2 + y^2 + z^2 = a^2$ and the plane $x + y + z = 0$.
- (d) $\int_C \vec{F} \cdot d\vec{r}$, where $\vec{F} = y\hat{i} + x\hat{j} + xyz^2\hat{k}$ and C is the circle: $x^2 - 2x + y^2 = 2, z = 1$.

5.2 Solutions

1. Evaluate the double integrals

- (a) $\iint_R x^2 dA$ where R is the region bounded by $y = x^2, y = x + 2$

The region is bounded by $y = x^2, y = x + 2$. Simple to notice that $y = x + 2$ is the upper curve and $y = x^2$ is the lower curve between the points of intersection $x = -1$ and $x = 2$. So it is enough to evaluate the iterated integral

$$\int_{x=-1}^2 \left(\int_{x^2}^{x+2} x^2 dy \right) dx = \int_{-1}^2 x^2(x+2-x^2)dx = \left[\frac{x^4}{4} + \frac{2x^3}{3} - \frac{x^5}{5} \right]_{-1}^2 = \frac{163}{60}$$

- (b) $\iint_R (x^2 + y^2) dA$ where $R : 0 \leq y \leq \sqrt{1-x^2}, 0 \leq x \leq 1$

The region is the part of the unit disc $x^2 + y^2 \leq 1$ that lies in the first quadrant. So the upper curve and lower curve can be easily identified as $y = \sqrt{1-x^2}$ and $y = 0$ respectively. So it is enough to evaluate

$$\int_{x=0}^1 \left(\int_0^{\sqrt{1-x^2}} (x^2 + y^2) dy \right) dx = \int_{\theta=0}^{\pi/2} \int_{r=0}^1 r^3 dr d\theta = \pi/8.$$

- (c) $\iint_R (a^2 - x^2 - y^2) dA$ where R is the region $x^2 + y^2 \leq a^2$.

Note that the upper curve is $y = \sqrt{a^2 - x^2}$ and the lower curve is $y = -\sqrt{a^2 - x^2}$ between $x = -a$ to $x = a$. Now it is easy to see that it is same as the iterated integral

$$I = \int_{x=-a}^a \left(\int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} (a^2 - x^2 - y^2) dy \right) dx.$$

Another way is to use the polar coordinates $x = r \cos \theta, y = r \sin \theta, 0 \leq r \leq a$ and $0 \leq \theta \leq 2\pi$. Then

$$I = \int_0^{2\pi} \int_0^a (a^2 - r^2) r dr d\theta = \frac{\pi a^4}{2}.$$

2. Evaluate the following integrals by changing the order of integration if needed:

- (a) $\int_0^3 \int_{-y}^y (x^2 + y^2) dx dy$

The region is bounded above by $y = 3$ and below by $x = -y$ from $x = -3$ to

$x = 0$ and $x = y$ from $x = 0$ to $x = 3$. So the integral may be written as

$$\int_{x=-3}^0 \left(\int_{y=-x}^3 (x^2 + y^2) dy \right) dx + \int_{x=0}^{x=3} \left(\int_{y=x}^{y=3} (x^2 + y^2) dy \right) dx$$

It is easy to evaluate the given integral instead of changing the order.

(b) $\int_0^\pi \int_0^\pi |\cos(x+y)| dx dy$

The integral may be split into two pieces where $\cos(x+y)$ is positive and negative respectively. In the domain bounded by $x = 0, y = 0$ and $x + y = \pi/2$ we have $\cos(x+y)$ is non-negative. In the domain bounded below by $x + y = \pi/2, (0 \leq x \leq \pi/2)$ and above by $y = \pi$, $\cos(x+y)$ is non-positive.

similarly, in the domain $\pi/2 \leq x \leq \pi, 0 \leq y \leq \frac{3\pi}{2} - x$, $\cos(x+y) \leq 0$ and in the domain $\pi/2 \leq x \leq \pi, \frac{3\pi}{2} - x \leq y \leq \pi$, $\cos(x+y) \geq 0$. Hence the given integral may be written as

$$\begin{aligned} \int_0^\pi \int_0^\pi |\cos(x+y)| dx dy &= \int_{x=0}^{\pi/2} \left(\int_{y=0}^{\frac{\pi}{2}-x} \cos(x+y) dy + \int_{\frac{\pi}{2}-x}^\pi -\cos(x+y) dy \right) dx \\ &\quad + \int_{x=\pi/2}^\pi \left(\int_0^{\frac{3\pi}{2}-x} -\cos(x+y) dy + \int_{\frac{3\pi}{2}-x}^\pi \cos(x+y) dy \right) dx \end{aligned}$$

(c) $\int_0^1 \int_2^{4-2x} dy dx$

The domain is bounded by $x = 0, y = 2$ and $y = 4 - 2x$. So it is easy to note that in the domain y ranges from 2 to 4 while x ranges from 0 to $4 - 2x$. So

$$\int_0^1 \int_2^{4-2x} dy dx = \int_{y=2}^4 \left(\int_{x=0}^{(4-y)/2} dx \right) dy$$

(d) $\int_0^{3/2} \int_0^{9-4x^2} 16x dy dx$

The domain is bounded by the parabola $y = 9 - 4x^2, x = 0$ and $y = 0$. Hence

$$\int_0^{3/2} \int_0^{9-4x^2} 16x dy dx = \int_{y=0}^9 \left(\int_{x=0}^{\frac{1}{2}\sqrt{9-y}} 16x dx \right) dy$$

(e) $\int_0^\pi \int_x^\pi \frac{\sin y}{y} dy dx$

The domain is bounded by the lines $x = 0, y = \pi$ and $y = x$. Hence

$$\int_0^\pi \int_x^\pi \frac{\sin y}{y} dy dx = \int_{y=0}^\pi \left(\int_{x=0}^y \frac{\sin y}{y} dx \right) dy$$

$$(f) \int_0^2 \int_0^{4-x^2} \frac{xe^{2y}}{4-y} dy dx$$

The domain is bounded by $x = 0, y = 0$ and $y = 4 - x^2$. Hence

$$\int_0^2 \int_0^{4-x^2} \frac{xe^{2y}}{4-y} dy dx = \int_0^4 \left(\int_0^{\sqrt{4-y}} \frac{xe^{2y}}{4-y} dx \right) dy$$

3. Find the volume of the following:

- (a) We use the formula $V = \iint_R f(x, y) dA$ where $f(x, y) \geq 0$ is a continuous real valued function defined over the domain R of the plane.

Here $f(x, y) = x^2 + y^2$ and the domain is bounded by $x = 0, y = 2 - x$ and $y = x$. So in this domain drawing a line parallel to y axis, it is easy to see that the domain may be described as $x \leq y \leq 2 - x, 0 \leq x \leq 1$. In other words, the upper curve is $y = 2 - x$ and lower curve is $y = x$ between $x = 0$ and $x = 1$. Hence

$$V = \iint_R (x^2 + y^2) dA = \int_0^1 \left(\int_x^{2-x} (x^2 + y^2) dy \right) dx = \frac{4}{3}$$

- (b) The domain R is bounded by $y = 2 - x^2$ and $y = x$ in the xy plane. The function $f(x, y) = x^2$. So the upper curve is $y = 2 - x^2$ and the lower curve is $y = x$. Hence

$$V = \iint_R x^2 dA = \int_{x=-2}^1 \left(\int_{y=x}^{2-x^2} x^2 dy \right) dx = \frac{63}{20}$$

- (c) The domain is bounded by $x = 0, y = 0$ and $x^2 + y^2 = 4$. The non-negative function is $f(x, y) = 3 - y$. The upper curve is $y = \sqrt{4 - x^2}$ and the lower curve is $y = 0$. Hence

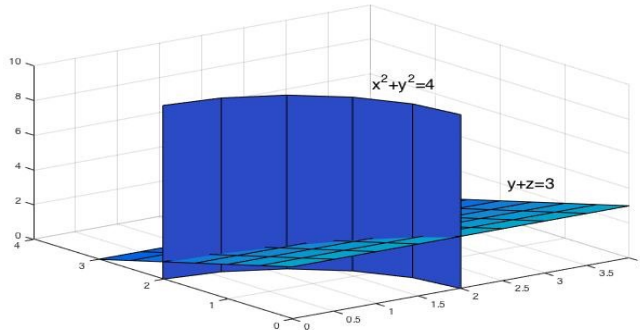


Figure 5: 3(c)

$$V = \iint_R (3 - y) dA = \int_0^2 \left(\int_0^{\sqrt{4-x^2}} (3 - y) dy \right) dx = \frac{9\pi - 8}{3}$$

- (d) The domain is bounded by $y = 2 - x$, $y = 0$ and $x = 0$. The non-negative function $f(x, y) = 12 - 3y^2$. Hence

$$V = \iint_R (12 - 3y^2) dA = \int_0^2 \left(\int_0^{2-x} (12 - 3y^2) dy \right) dx = 20$$

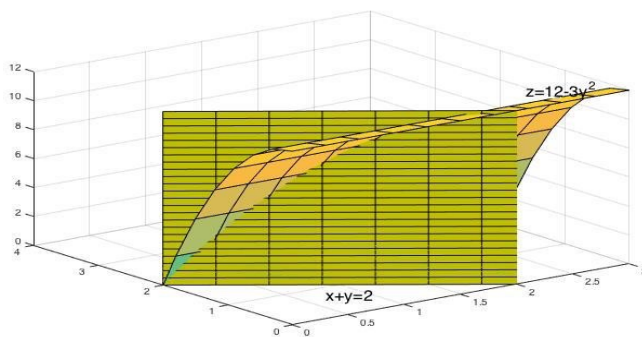


Figure 6: 3(d)

- (e) The non-negative function is $f(x, y) = 1 + x - y$ over the domain R bounded by $y = 0$, $x = 0$ and $y = 1 + x$. Hence

$$V = \iint_R (1 - y + x) dA = \int_{x=-1}^0 \left(\int_0^{1+x} (1 - y + x) dy \right) dx = \frac{1}{6}$$

4. Use the given transformations to transform the integrals and evaluate them:

- (a) Here we get $x = \frac{2u-v}{5}$ and $y = \frac{3v-u}{10}$. The Jacobian of the transformation is

$$J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} 2/5 & -1/5 \\ -1/10 & 3/10 \end{vmatrix} = 1/10$$

The image is again a rectangle with sides $u = 2$, $u = 6$, $v = 0$ and $v = 4$. Hence

$$\iint_R (3x^2 + 14xy + 8y^2) dA = \int_{u=2}^6 \left(\int_0^4 uv dv \right) du = \frac{64}{5}$$

- (b) The Jacobian $J = -\frac{1}{3}$ and $|J| = 1/3$. The given domain is the triangle bounded by $y = x$, $y = 0$ and $x + 2y = 2$. The image of this triangle under the transformation is again a triangle bounded by $v = 0$, $v = u$ and $u = 2$. Hence

$$\int_0^{2/3} \int_y^{2-2y} (x + 2y) e^{(y-x)} dA = \int_{u=0}^2 \left(\int_0^u u e^{-v} \frac{1}{3} dv \right) du = \frac{1}{3} (3e^{-2} + 1)$$

- (c) The domain of integration is the domain bounded by the hyperbolas $xy = 1$, $xy = 2$, $x^2 - y^2 = 1$ and $x^2 - y^2 = 2$. Under the given transformations this hyperbolas are transformed into the lines $u = 1$, $u = 2$, $v = 1$ and $v = 2$. The Jacobian J may be calculated using the relation $JJ^{-1} = 1$.

$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = -2(x^2 + y^2).$$

Therefore

$$\iint_R (x^2 + y^2) dA = \int_{u=1}^2 \left(\int_{v=1}^2 (x^2 + y^2) \cdot \frac{1}{2(x^2 + y^2)} dv \right) du = \int_{u=1}^2 \int_{v=1}^2 \frac{1}{2} dv du = 2.$$

5. The sides of the parallelogram are $x + y = 1$, $x + y = 4$, $x - 2y = 1$ and $x - 2y = -1$. So it is convenient to consider the transformation $u = x + y$ and $v = x - 2y$. So the inverse of the transformation is $3x = v + 2u$ and $3y = u - v$. The Jacobian is $-\frac{1}{3}$. Hence

$$\iint_R (x + y)^2 dA = \int_{u=1}^4 \int_{v=-1}^1 u^2 \frac{4}{3} dv du = 56.$$

6. (a) Area of the region R in the polar coordinates is given by the formula $A = \iint_R r dr d\theta$. By drawing a ray from origin, it is easy to see that the ray intersects the domain R at $r = 1$ (closest from origin) and $r = 1 + \cos \theta$ (farthest from origin). Hence the area is

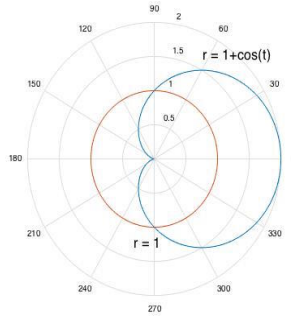


Figure 7: 6(a)

$$A = \int_{\theta=0}^{\pi/2} \left(\int_1^{1+\cos \theta} r dr \right) d\theta = \frac{8 + \pi}{4}.$$

- (b) The domain is symmetric with respect to x axis and y axis. So it is enough to evaluate the area in the first quadrant. Here as above we can see that the region is bounded by $r = 0$ and $r = 1 - \cos \theta$. Hence

$$A = 4 \int_0^{\pi/2} \left(\int_0^{1-\cos \theta} r dr \right) d\theta = \frac{3\pi}{2}$$

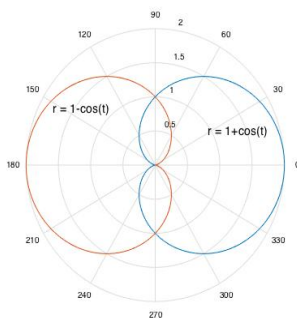


Figure 8: 6(b)

7. (a) As in problem 3, we use the formula $V = \iint_R f(x, y) dA$, where $f(x, y) \geq 0$. In this case $f(x, y) = x$ and the domain R is bounded from below by $r = 1$ and above by $r = 1 + \cos \theta$ (with respect to origin). Hence

$$V = 2 \int_0^{\pi/2} \int_1^{1+\cos \theta} (r \cos \theta) r dr d\theta = \frac{4}{3} + \frac{5\pi}{8}$$

(b)

$$\begin{aligned} \text{Volume} &= \iint_R (\sqrt{2-r^2}) r dr d\theta \\ &= 4 \int_{\theta=0}^{\pi/4} \left(\int_0^{\sqrt{2\cos 2\theta}} \sqrt{2-r^2} r dr \right) d\theta \\ &= \frac{2\pi\sqrt{2}}{3} - \frac{32}{3} \int_0^{\pi/4} (1 - \cos^2 \theta) \sin \theta d\theta \\ &= \frac{6\pi\sqrt{2} + 40\sqrt{2} - 64}{9} \end{aligned}$$

8. (a) Using the cylindrical coordinates $x = r \cos \theta$, $y = r \sin \theta$ and $z = z$, the given domain may be represented as

$$D = \{(r, \theta, z) : 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi, -1 \leq z \leq 1\}.$$

Hence the integral becomes

$$\begin{aligned} \iiint_D z^2(x^2 + y^2) dV &= \int_{z=-1}^1 \int_{\theta=0}^{2\pi} \int_{r=0}^1 (z^2 r^2) r dr d\theta dz \\ &= \int_{z=-1}^1 \int_0^{2\pi} z^2 \frac{1}{4} d\theta dz = \frac{\pi}{3} \end{aligned}$$

- (b) Again using the cylindrical coordinates

$$D = \{(r, \theta, z) : 0 \leq z \leq r^2, 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi\}.$$

Hence

$$\iiint_D xyz dV = \int_{\theta=0}^{2\pi} \int_{r=0}^1 \int_{z=0}^{r^2} zr^3 \cos \theta \sin \theta dz dr d\theta = 0$$

- (c) In this case, using the spherical coordinates $x = r \cos \theta$, the domain is represented as

$$D = \{(r, \theta, \phi) : 0 \leq r \leq 1, 0 \leq \phi \leq \pi, 0 \leq \theta \leq 2\pi\}$$

Hence

$$\iiint_D e^{(x^2+y^2+z^2)^{3/2}} dV = \int_{\theta=0}^{2\pi} \int_0^\pi \int_0^1 e^{r^3} r^2 \sin \phi dr d\phi d\theta = \frac{4\pi}{3}(e-1)$$

9. (a) This is like tetrahedron with base on the xz -plane. The limits of y for the solid are $y = 0$ to $y = 2 - 2z$. The triangle on the xz -plane is bounded by $x = 0, y = 0$ and $x + z = 1$. On this triangle $0 \leq z \leq 1 - x$ and $0 \leq x \leq 1$. Hence volume is equal to the iterated integral

$$V = \int_{x=0}^1 \int_{z=0}^{1-x} \int_{y=0}^{2-2z} dy dz dx$$

- (b) Imagine the solid as the piece of parabolic cylinder on over the region R of the xy -plane bounded by $x = 4 - y^2$ in the first quadrant. Over the solid z varies from 0 to $2 - y$ and $R = \{(x, y) : 0 \leq y \leq \sqrt{4 - x}, 0 \leq x \leq 4\}$. Hence

$$V = \int_{x=0}^4 \int_{y=0}^{\sqrt{4-x}} \int_0^{2-y} dz dy dx = \int_{y=0}^2 \int_{x=0}^{4-y^2} \int_0^{2-y} dz dx dy$$

$$(c) V = \int_{x=0}^1 \int_0^{2-2x} \int_0^{3-3x-\frac{3y}{2}} dz dy dx$$

$$(d) V = 8 \int_{x=0}^1 \int_{y=0}^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2}} dz dy dx$$

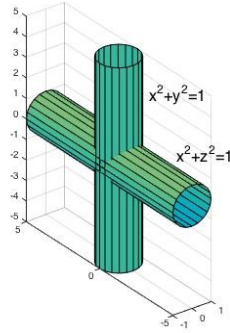


Figure 9: 9(d)

$$(e) V = \int_{x=-2}^2 \int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_0^{3-x} dz dy dx$$

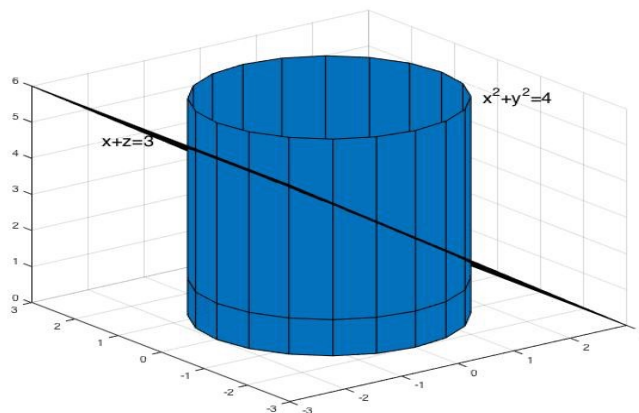


Figure 10: 9(e)

$$(f) \quad V = \int_{x=-1}^2 \int_{y=x^2}^{x+2} \int_{z=(x^2+y^2)/4}^{x+3} dz dy dx = \frac{783}{70}$$

- (g) The region is bounded above by the surface $z^2 = 2 - x^2 - y^2$ and the lower surface $z = x^2 + y^2$. Hence the limits of z are $x^2 + y^2 \leq z \leq 2 - x^2 - y^2$.

The given surfaces intersect on the plane $z = 1$ ($z^2 + z - 2 = 0 \implies z = 1, -2$). So the solid base is the circle $x^2 + y^2 = 1$. The limits of y are $-\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}$ and that of x are $-1 \leq x \leq 1$.

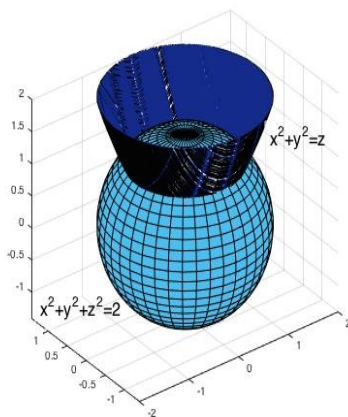


Figure 11: 9(g)

Now going to cylindrical coordinates

$$V = 4 \int_{\theta=0}^{\pi/2} \int_{r=0}^1 \int_{r^2}^{\sqrt{2-r^2}} r \, dz dr d\theta$$

- (h) In cylindrical coordinates, the given solid is bounded by $z = r$ and $z = r^2$. The solid is obtained by rotating the area between $z = r$ and $z = r^2$. Hence

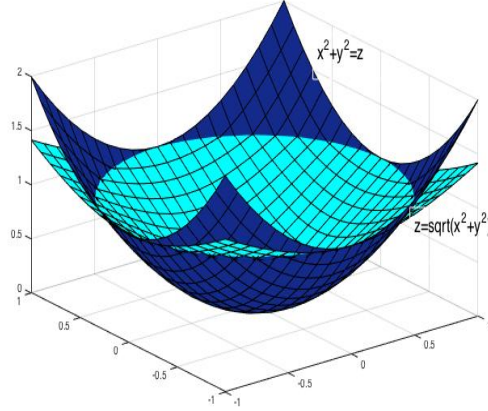


Figure 12: 9(h)

$$V = \int_{\theta=0}^{2\pi} \int_{r=0}^1 \int_{z=r^2}^r r dz dr d\theta = \frac{\pi}{6}$$

10. Mass $M = \iiint_D \delta(x, y, z) dV = \int_0^1 \int_0^1 \int_0^1 (x + y + z + 1) dz dy dx$

First moments about the coordinate planes:

$$M_{yz} = \iiint_D x \delta(x, y, z) dV = \int_0^1 \int_0^1 \int_0^1 x(x + y + z + 1) dz dy dx$$

$$M_{xz} = \iiint_D y \delta(x, y, z) dV, \quad M_{xy} = \iiint_D z \delta(x, y, z) dV$$

$$\text{Center of mass: } \bar{x} = \frac{M_{yz}}{M}, \quad \bar{y} = \frac{M_{xz}}{M}, \quad \bar{z} = \frac{M_{xy}}{M}$$

Moments of inertia about the coordinate axes:

$$I_x = \iiint_D (y^2 + z^2) \delta(x, y, z) dV, \quad I_y = \iiint_D (x^2 + z^2) \delta(x, y, z) dV, \quad I_z = \iiint_D (x^2 + y^2) \delta(x, y, z) dV$$

11. Use the formulas above in the domain $D = \{(x, y, z) : 0 \leq x \leq 2, 0 \leq y \leq x^2, 0 \leq z \leq 4 - x^2\}$.

12. (a) The Jacobian of the transformation is

$$J(u, v, w) = \frac{\partial(x, y, z)}{\partial(u, v, w)} \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc.$$

The transformation maps the ellipsoid into the unit ball $u^2 + v^2 + w^2 \leq 1$. Hence

$$V = \iiint_{\text{ellipsoid}} dV = \iiint_{\text{unit ball}} abcdV = \frac{4\pi}{3} abc.$$

- (b) The inverse of the given transformation is $x = u, y = \frac{v}{u}$ and $z = \frac{w}{3}$. So the required Jacobian is

$$J = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} 1 & 0 & 0 \\ -\frac{v}{u^2} & \frac{1}{u} & 0 \\ 0 & 0 & \frac{1}{3} \end{vmatrix} = \frac{1}{3u}$$

and the given integral is

$$\iiint_D (x^2y + 3xyz) dV = \frac{1}{3} \int_{w=0}^3 \int_{v=0}^2 \int_{u=1}^2 (v + \frac{vw}{u}) du dv dw = 2 + 3 \log 2$$

13. (a) The surface S may be defined as $f(x, y, z) = z - g(x, y) = 0$ over R of xy -plane. So we can project it on to R and the normal to R is $\hat{k}$. We use the formula

$$\text{Surface area of } S = \iint_R \frac{|\nabla f|}{|\nabla f \cdot \hat{p}|} dA$$

where $\hat{p}$ is the unit normal to the R , which is $\hat{k}$ in this case. The surface S is $f(x, y, z) = x^2 + y^2 - z$, $\nabla f = 2x\hat{i} + 2y\hat{j} - \hat{k}$ and $|\nabla f| = \sqrt{4x^2 + 4y^2 + 1}$. The domain $R = \{(x, y) : x^2 + y^2 \leq 2\}$ and $|\nabla f \cdot \hat{p}| = 1$. Hence

$$S = \iint_R \sqrt{4x^2 + 4y^2 + 1} dA = \int_0^{2\pi} \int_0^{\sqrt{2}} (\sqrt{4r^2 + 1}) r dr d\theta = \frac{13\pi}{3}$$

- (b) The required surface is on the plane $x + 2y + 2z = 5$ and it is defined on as $z - g(x, y) = 0$ on R of xy -plane that is bounded by $x = y^2$ and $x = 2 - y^2$.

The equation of surface is $f(x, y, z) = x + 2y + 2z - 5 = 0$. So

$$\nabla f = \hat{i} + 2\hat{j} + 2\hat{k}, \hat{p} = \hat{k}, |\nabla f| = 3, |\nabla f \cdot \hat{p}| = 2.$$

Hence

$$S = \iint_R \frac{3}{2} dA = \frac{3}{2} \int_{-1}^1 \int_{y^2}^{2-y^2} dx dy = 4.$$

- (c) The surface S is the portion of the sphere above xy -plane. So the equation of surface is $f(x, y, z) = x^2 + y^2 + z^2 - 2 = 0$. The cone $z = \sqrt{x^2 + y^2}$ and sphere $x^2 + y^2 + z^2 = 2$ intersect on $z = 1$ (by solving $z^2 + z - 2 = 0$). So $\hat{p} = \hat{k}$, $|\nabla f| = \sqrt{8}$, $|\nabla f \cdot \hat{p}| = 2z$. The projection of this portion of the surface that lies on $z = 1$ onto xy -plane is the unit disc $x^2 + y^2 \leq 1$. Therefore, $R = \{(x, y) : x^2 + y^2 \leq 1\}$. Also

$$\frac{|\nabla f|}{|\nabla f \cdot \hat{p}|} = \frac{\sqrt{8}}{2z} = \frac{\sqrt{2}}{\sqrt{2 - (x^2 + y^2)}}$$

as $x^2 + y^2 + z^2 = 2$ on S . Hence

$$S = \iint_R \frac{\sqrt{2}}{\sqrt{2 - (x^2 + y^2)}} dA = \sqrt{2} \int_0^{2\pi} \int_0^1 \frac{r dr d\theta}{\sqrt{2 - r^2}} = 2\pi(2 - \sqrt{2})$$

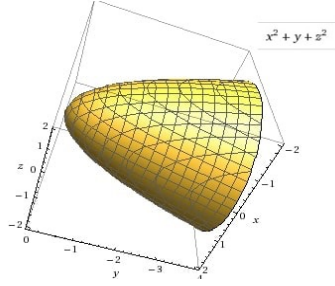


Figure 13: 13(d)

- (d) In this case, $\hat{p} = \hat{k}$, $f(x, y, z) = x^2 + y + z^2 - 2 = 0$ and $R = \{(x, z) : x^2 + z^2 \leq 2\}$.
Hence

$$S = \iint_R \sqrt{4x^2 + 4z^2 + 1} dA = \int_0^{2\pi} \int_0^{\sqrt{2}} \sqrt{4r^2 + 1} r dr d\theta = \frac{13\pi}{3}$$

- (e) The surface is symmetric and its area is equal to 4 times the surface area of $f(x, y, z) = y^2 + z^2 - 4 = 0$ defined over $x^2 + y^2 \leq 4$. $\nabla f = 2y\hat{j} + 2z\hat{k}$, $\hat{p} = \hat{k}$ and $\nabla f \cdot \hat{p} = 2z = 2\sqrt{4 - y^2}$ on S . Hence the surface area is

$$S = 4 \iint_R \frac{2dA}{\sqrt{4 - y^2}} = 8 \int_{-2}^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \frac{dx dy}{\sqrt{4 - y^2}}$$

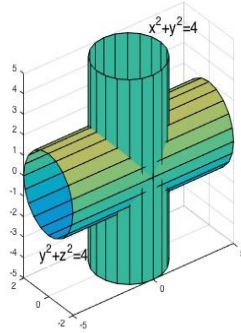


Figure 14: 13(e)

- (f) The equation of surface is $x^2 + y^2 - z^2 = 0$. The projection of the required surface onto xy -plane is $(x - 2)^2 + y^2 \leq 4$. On the surface $|\nabla f \cdot \hat{p}| = 2z$ and $|\nabla f| = 2\sqrt{2}z$.
Hence

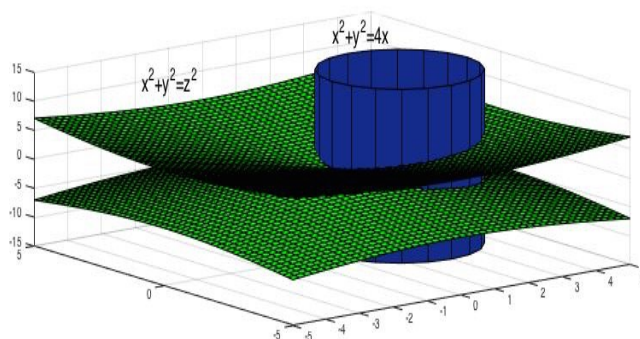


Figure 15: 13(f)

$$S = \iint_R \frac{|\nabla f|}{|\nabla f \cdot \hat{p}|} dA = 4\sqrt{2}\pi$$

(g) The parametrization of the surface using cylindrical coordinates is

$$\vec{r}(\theta, z) = 4\cos^2\theta \hat{i} + 2\sin 2\theta \hat{j} + z\hat{k}, 0 \leq z \leq 4\cos\theta, -\pi/2 \leq \theta \leq \pi/2.$$

Limits of z are obtained by substituting $x = 4\cos^2\theta, y = 2\sin 2\theta$ in $z^2 = x^2 + y^2$.

$$\vec{r}_\theta \times \vec{r}_z = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -4\sin 2\theta & 4\cos 2\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = 4\cos 2\theta \hat{i} + 4\sin 2\theta \hat{j}.$$

Hence

$$S = \int_{\theta=-\pi/2}^{\pi/2} \int_{z=0}^{4\cos\theta} 4dz d\theta$$

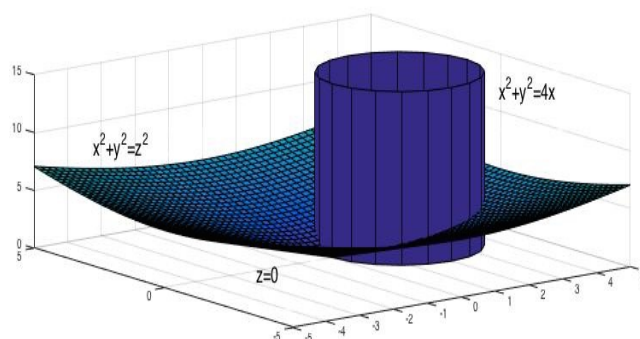


Figure 16: 13(g)

(h) The parametrization of the surface using cylindrical coordinates is

$$\vec{r}(\theta, z) = a \cos \theta \hat{i} + a \sin \theta \hat{j} + z \hat{k}$$

over the parameter domain

$$R = \{(\theta, z) : -\pi/2 \leq \theta \leq \pi/2, 0 \leq z \leq 2a \cos \theta\}.$$

Therefore,

$$\vec{r}_\theta \times \vec{r}_z = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -a \sin \theta & a \cos \theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = a \cos \theta \hat{i} + a \sin \theta \hat{j}.$$

Hence

$$S = \int_{-\pi/2}^{\pi/2} \int_0^{2a \cos \theta} |\vec{r}_\theta \times \vec{r}_z| dz d\theta = 4a^2$$

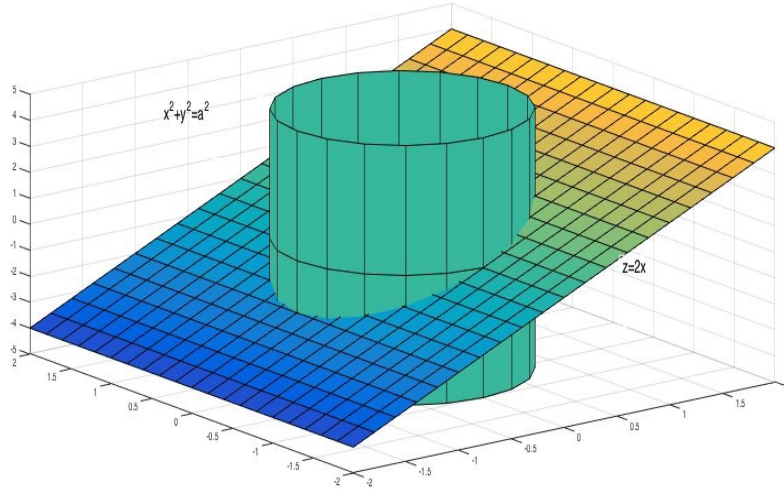


Figure 17: 13(h)

(i) The surface is given by $y = \sqrt{2x - x^2} = 0$ which is symmetric about zx -plane. $\hat{p} = \hat{j}$, $|\nabla f \cdot \hat{p}| = 2y = 2\sqrt{2x - x^2}$ on S and $|\nabla f| = \sqrt{4(x-1)^2 + 4y^2} = 2$. The projection on the zx -plane is

$$R = \{(z, x) : 0 \leq x \leq 2, 0 \leq z \leq 4 - 2x\}.$$

Hence

$$S = \iint_R \frac{dA}{\sqrt{2x - x^2}} = \int_{x=0}^2 \int_{z=0}^{4-2x} \frac{dz dx}{\sqrt{2x - x^2}} = 4\pi$$

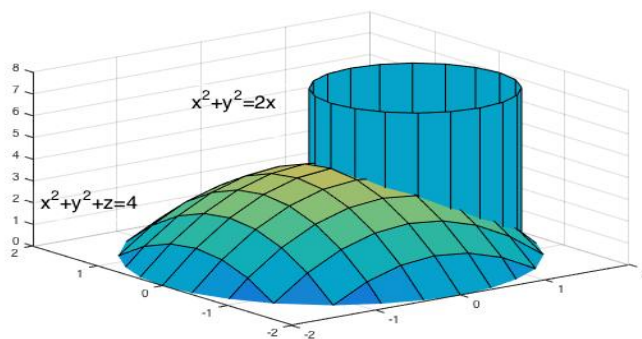


Figure 18: 13(i)

14. (a) The bottom face S_1 of the cube is in the xy -plane, so the surface is defined by $f(x, y, z) = z = 0$. On this face, $g(x, y, z) = x + y$ and $\hat{p} = \hat{k}$. $|\nabla f| = \hat{k}$, and $|\nabla f \cdot \hat{p}| = 1$. Hence

$$\iint_{S_1} g dS = \iint_R (x + y) dx dy = \int_0^a \int_0^a (x + y) dx dy = a^3.$$

Because of symmetry, we also get a^3 over the face of the cube in the xz -plane and yz -plane. Next, on the top of the cube $g = x + y + a$ and $f = z = a$ and $\iint_{S_2} g dS = 2a^3$. Again by symmetry, $\iint_{\text{cube}} g dS = 9a^3$.

- (b) On the face S_1 in the xz -plane, we have $S_1 : f(x, y, z) = y = 0$ and $g(x, y, z) = g(x, 0, z) = z$, $\hat{p} = \hat{j}$. $\nabla f = \hat{j}$, $|\nabla f \cdot \hat{p}| = 1$ and hence $\iint_{S_1} g dS = \int_0^1 \int_0^2 z dx dy = 1$. On the face in the xy -plane, we have $S_2 : f(x, y, z) = z = 0$, $g(x, y, z) = y$, $\hat{p} = \hat{k}$ and $\nabla f = \hat{k}$ and hence $\iint_{S_2} g dS = \int_0^1 \int_0^2 y dx dy = 1$.

On the triangular face in the plane $x = 2$, $S_3 : f(x, y, z) = x - 2 = 0$ and $g(x, y, z) = y + z$. Hence $\iint_{S_3} g dS = \iint_{S_3} (y + z) dS = \int_0^1 \int_0^{1-y} (y + z) dz dy = 1/3$

On the triangular face in the yz plane, $f = x = 0$, $g = y + z$, $\hat{p} = \hat{i}$ and $\iint_{S_4} g dS = \int_0^1 \int_0^{1-y} (y + z) dz dy = 1/3$.

On the sloped face, $f = y + z - 1$, $g = y + z$, $\hat{p} = \hat{k}$ and $\nabla f = \hat{j} + \hat{k}$ and hence $\iint_{S_5} g dS = \int_0^1 \int_0^2 \sqrt{2} dx dy = 2\sqrt{2}$

- (c) On the coordinate planes, $g(x, y, z) = 0$ and the integral over these faces is zero. On the face $S_1 : x = a$, $f = x - a$, $g = ayz$, $\hat{p} = \hat{i}$ and $\nabla f = \hat{i}$. Therefore, $\iint_{S_1} g dS = a \int_0^c \int_0^b yz dy dz = \frac{ab^2c^2}{4}$. On the face $S_2 : y = b$, $f = y - b = 0$, $g = bxz$, $\hat{p} = \hat{j}$, $\nabla f = \hat{j}$ and $\iint_{S_2} g dS = \frac{a^2bc^2}{4}$. On the face $S_3 : z = c$, $f = z - c = 0$, $g = cxy$, $\hat{p} = \hat{k}$, $\nabla f = \hat{k}$ and $\iint_{S_3} g dS = \frac{a^2b^2c}{4}$.

15. A parametrization of C is $x = 5 + 2 \cos t$, $z = 7 + 2 \sin t$, $0 \leq t \leq 2\pi$. Hence

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} (-56 \sin t - 16 \sin^2 t - 30 \cos t - 12 \cos^2 t) dt = -28\pi$$

16. $x = 4 \cos t, y = 3 \sin t$ and the given curve is $\vec{r}(t) = 4 \cos t \hat{i} + 3 \sin t \hat{j}, 0 \leq t \leq 2\pi$. Now substituting in F , we see $\vec{F} \cdot d\vec{r} = -15 \sin 2t + 48$. Therefore $W = \int_0^{2\pi} (-15 \sin 2t + 48) dt = 96\pi$.

17. Work done $W = \int_{(-1,0)}^{(1,0)} \vec{F} \cdot d\vec{r}$. The ellipse is $C : x = \cos t, y = b \sin t$. As we move from $(-1, 0)$ to $(1, 0)$, t varies from π to 0 . Therefore

$$W = \int_C \vec{F} \cdot d\vec{r} = - \int_0^\pi (3b^2 \sin^2 t + 2)(-\sin t) + 16b \cos^2 t dt = 4b^2 - 8\pi b + 4$$

Now by the derivative test, we can show that W is minimum at $t = \pi$.

18. (a) By Green's theorem $\oint_C M dx + N dy = \iint_D (\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}) dA$. The region D is the domain bounded by the curve C . Here $M = y^2$ and $N = x^2$. Hence

$$\iint_D 2(x - y) dA = 2 \int_0^1 \int_0^{1-x} (x - y) dy dx = 0.$$

- (b) In this case $M = 3y$ and $N = 2x$. Therefore,

$$\iint_D (2 - 3) dA = - \int_0^\pi \int_0^{\sin x} dx dy = -2.$$

- (c) Here $M = 6y + x$ and $N = y + 2x$. Therefore,

$$\iint_D (2 - 6) dA = -16\pi.$$

19. Use the formula: Area of $R = \frac{1}{2} \oint_C x dy - y dx$, where R is the region bounded by the curve C

(a) $R = \frac{1}{2} \int_0^{2\pi} a^2 \cos^2 t + a^2 \sin^2 t = a^2 \pi$.

(b) $R = \frac{1}{2} \int_0^{2\pi} 3 \cos^4 t \sin^2 t + 3 \cos^2 t \sin^4 t = \frac{3\pi}{8}$

(c) $R = \frac{1}{2} \int_{-\sqrt{3}}^{\sqrt{3}} t^2(t-1) - 2t(\frac{t^3}{3} - t) 2t = \frac{8}{5}\sqrt{3}$

20. (a) $M = 4x - 2y, N = 2x - 4y$ and $x = a \cos t, y = a \sin t, 0 \leq t \leq 2\pi$. The work done W is

$$W = \oint_C (4x - 2y) dx + (2x - 4y) dy = 4 \iint_R dA = 16\pi$$

- (b) In this case, R is the domain bounded by ellipse

$$W = \iint_R (y - 1) dy dx = -6\pi$$

21. (a) The projection of the surface S on the xy -plane is the rectangle R with vertices $(1, -1, 0), (1, 1, 0), (0, 1, 0)$ and $(0, -1, 0)$. The surface is defined with equation $f(x, y, z) = y^2 + z^2 - 1 = 0$. So $\nabla f = 2y\hat{j} + 2z\hat{k}$ and $|\nabla f| = \sqrt{4(y^2 + z^2)} = 2$ on S . Therefore, unit outward normal is $\hat{n} = \frac{\nabla f}{|\nabla f|} = y\hat{j} + z\hat{k}$, and $\hat{p} = \hat{k}$. Also $\frac{|\nabla f|}{|\nabla f \cdot \hat{k}|} = \frac{1}{z}$. Hence

$$\text{Flux} = \iint_S \vec{F} \cdot \hat{n} dS = \iint_R \frac{1}{z} z dA = \text{Area of } R = 2$$

- (b) The surface (cap) is defined with equation $S : f(x, y, z) = x^2 + y^2 + z^2 - 25 = 0$ and the unit normal is $\hat{n} = \frac{1}{5}(x\hat{i} + y\hat{j} + z\hat{k})$ and $\vec{F} \cdot \hat{n} = \frac{1}{5}(x^2z + y^2z + z)$. The projection of S on xy -plane is $x^2 + y^2 \leq 16$ and $\hat{p} = \hat{k}$. Hence

$$\begin{aligned}\iint_S \vec{F} \cdot \hat{n} dS &= \iint_R \vec{F} \cdot \hat{n} \frac{|\nabla f|}{|\nabla f \cdot \hat{n}|} dx dy \\ &= \iint_R \frac{z}{5}(x^2 + y^2 + 1) \frac{5}{z} dA = \int_0^{2\pi} \int_0^4 (r^2 + 1) r dr d\theta = 144\pi\end{aligned}$$

Across the bottom of the cap: $S_1 : f(x, y, z) = z - 3, \hat{p} = \hat{k}, \vec{F} \cdot \hat{n} = -1, \hat{n} = -\hat{k}$. The flux is $\iint_{S_1} \vec{F} \cdot \hat{n} dS = \iint_R (-1) dx dy = -16\pi$. The total flux is 128π .

- (c) We use simple parametrization of the surface $\vec{r}(x, z) = x\hat{i} + x^2\hat{j} + z\hat{k}, 0 \leq x \leq 1, 0 \leq z \leq 4$.

$$\vec{r}_x \times \vec{r}_z = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2x & 0 \\ 0 & 0 & 1 \end{vmatrix} = 2x\hat{i} - \hat{j}$$

and the normal is

$$\hat{n} = \frac{\vec{r}_x \times \vec{r}_z}{|\vec{r}_x \times \vec{r}_z|} = \frac{1}{\sqrt{4x^2 + 1}}(2x\hat{i} - \hat{j}).$$

On the surface S , $\vec{F} \cdot \hat{n} = \frac{2x^3z - x}{\sqrt{4x^2 + 1}}$. Hence

$$\begin{aligned}\text{Flux} &= \iint_S \vec{F} \cdot \hat{n} dS = \iint_S \vec{F} \cdot \hat{n} |\vec{r}_x \times \vec{r}_z| dx dz \\ &= \int_0^4 \int_0^1 (2x^3z - x) dx dz = 2\end{aligned}$$

- (d) Take the equation of surface $S : f(x, y, z) = x + y + z = 4$ and its projection R on the xy -plane as the given rectangle. $\hat{n} = \frac{\nabla f}{|\nabla f|}$ and $\hat{p} = \hat{k}$. Use the similar calculations as above and substitute in

$$\text{Flux} = \iint_S \vec{F} \cdot \hat{n} dS = \iint_R \vec{F} \cdot \hat{n} \frac{|\nabla f|}{|\nabla f \cdot \hat{p}|} dx dy$$

- (e) The given surface is the surface of ellipsoid $S : f(x, y, z) = x^2 + 5y^2 + (z - 3)^2 - 16 = 0$ centered at $(0, 0, 3)$. Use the method described above with given $\vec{F}$.

22. (a) The curve C may be parametrized as $\vec{r}(t) = \cos t\hat{i} + \sin t\hat{j} + t\hat{k}, 0 \leq t \leq \pi/2$ and $\vec{r}' \cdot d\vec{r}' = -\sin t \cos t + \sin t \cos t + t = t$

$$\int_C \vec{r}'(t) \cdot d\vec{r}' = \int_0^{\pi/2} t dt = \frac{\pi^2}{8}$$

- (b) The curve is C is $\vec{r}(t) = \cos t\hat{i} + \sin t\hat{j}, 0 \leq t \leq \pi/2$ and $\vec{v} \cdot d\vec{r}' = -\sin^2 t + 2\cos^2 t$.

$$\int_C \vec{v} \cdot d\vec{r}' = \int_0^{\pi/2} (2\cos^2 t - \sin^2 t) dt = \pi/4$$

- (c) This problem can be solved directly also. Here we give a solution using concepts from Linear algebra. We define the new coordinate system in which the plane $x + y + z = 0$ is the XY plane.

Define the new coordinate axes as

$$X = \frac{1}{\sqrt{3}}(x + y + z), Y = \frac{1}{\sqrt{2}}(x - y), Z = \frac{1}{\sqrt{6}}(x + y - 2z).$$

The given curve in the new coordinate system is

$$X = 0, Y = a \cos t, Z = a \sin t, 0 \leq t \leq 2\pi.$$

Also

$$x = \frac{1}{\sqrt{3}}X + \frac{1}{\sqrt{2}}Y + \frac{1}{\sqrt{6}}Z, y = \frac{1}{\sqrt{3}}X - \frac{1}{\sqrt{2}}Y + \frac{1}{\sqrt{6}}Z, z = \frac{1}{\sqrt{3}}X - \frac{2}{\sqrt{6}}Z.$$

Then

$$\int_0^{2\pi} (y - z)dx = \int_0^{2\pi} \left(-\frac{a}{\sqrt{2}} \cos t + \frac{3a}{\sqrt{6}} \sin t\right) \left(-\frac{a}{\sqrt{2}} \sin t + \frac{a}{\sqrt{6}} \cos t\right) dt = -\frac{2\pi a^2}{\sqrt{3}}$$

Similarly,

$$\int_C (x + z)dy = 0, \text{ and } \int_C (x - y)dz = -\frac{2\pi a^2}{\sqrt{3}}.$$

Hence

$$\int_C (y - z)dx + (x + z)dy + (x - y)dz = -\frac{4\pi a^2}{\sqrt{3}}$$

- (d) The given circle C is: $(x - 1)^2 + y^2 = 3$, $z = 1$ and so on the circle $dz = 0$.
Therefore,

$$\begin{aligned} \oint_C \vec{F} \cdot d\vec{r} &= \oint_C F_1 dx + F_2 dy + F_3(0) \\ &= \oint_C y dx + x dy = \oint_C \nabla(xy) \cdot d\vec{r} = 0, \end{aligned}$$

since C is closed.

References

1. Thomas' Calculus, Chapters 15, 16
2. Advanced engineering Mathematics, E.Kreyszig, Chapter 9