Jan. 10

Borel σ-field

Def: Let Ω be a non-empty set and $\mathcal{A}$ be the collection of open subsets of Ω. If $\sigma(\mathcal{A})$ is the σ-field generated by $\mathcal{A}$, then $\sigma(\mathcal{A})$ is called Borel σ-field on Ω.

Remark:

(I) Banach - Tarski Paradox

(II) Vitali sets

Eg: $\mathbb{R}$ is set of real numbers

$$\mathcal{A} = \{(a,b) : -\infty < a \leq b < \infty\}$$

then

$\sigma(\mathcal{A})$ is the Borel σ-field on $\mathbb{R}$.

Remark: $\mathbb{B}(\mathbb{R})$ will denote Borel σ-field on $\mathbb{R}$.

Eg: Assume the probability space $(\Omega, \mathcal{F}, \mathbb{P})$

where

$$\Omega = (0,1)$$

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$\mathcal{F} = \mathcal{B}((0,1))$

$$\boxed{\mathbb{P}(A) = \int_{A \cap (0,1)} dx}$$

if $0 \leq a \leq b < 1$

Then $\mathbb{P}((a,b)) = b - a$

<u>Remark:</u> Lebesgue integral

$$\int_{\mathbb{Q} \cap \mathbb{R}} dx = 0$$

$\mathbb{Q}$ is set of rational numbers.

$$\mathbb{P}(\mathbb{Q} \cap (0,1))$$
$$= \int_{\mathbb{Q} \cap (0,1)} dx = 0$$

Properties of probability measure

Let $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space. then

(1) If $A \in \mathcal{F}$ then

$$\mathbb{P}(A) + \mathbb{P}(A^c) = 1$$

(ii) If $A, B \in \mathcal{F}$ then

(ii) If $A, B \in \mathcal{G}$ then

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$$

Proof: Let $A_1, A_2 \in \mathcal{G}$ and $A_1 \cap A_2 = \phi$.

Then

$$\mathbb{P}(A_1 \cup A_2) = \mathbb{P}(A_1) + \mathbb{P}(A_2)$$

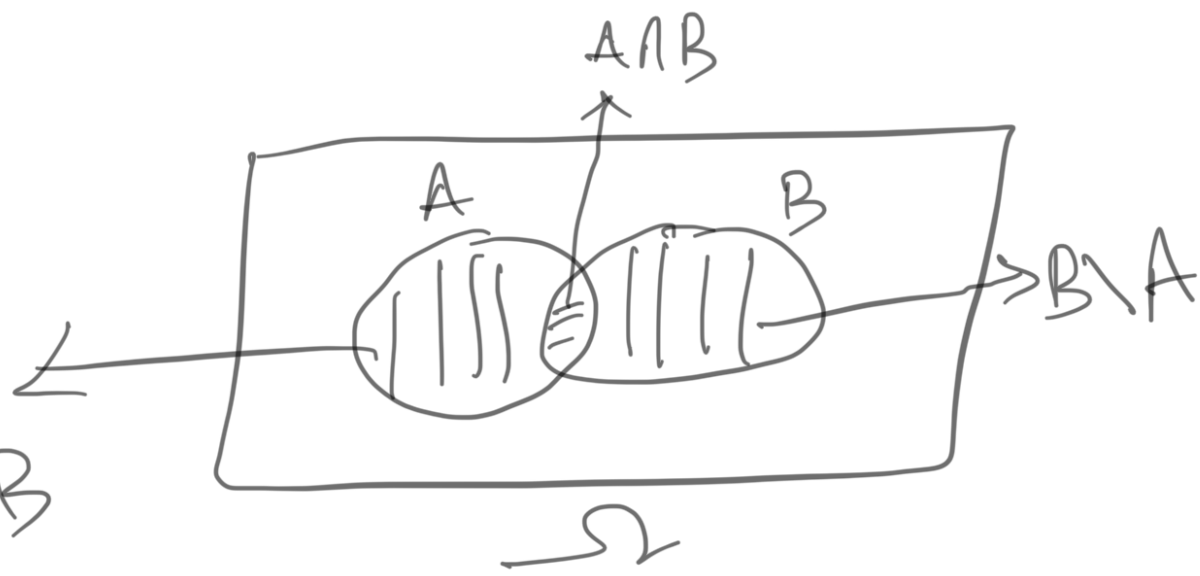
(1) Take $A_1 = A$

$A_2 = A^c$

$$\mathbb{P}(A \cup A^c) = \mathbb{P}(A) + \mathbb{P}(A^c)$$

$$\Rightarrow \mathbb{P}(\Omega) = \mathbb{P}(A) + \mathbb{P}(A^c)$$

(ii)



$$A \cup B = (A\setminus B) \cup (B\setminus A) \cup (A\cap B)$$

Clearly from venn-diagram

$A\setminus B$, $A\cap B$, $B\setminus A$ are disjoint sets.

Using third axiom

$$\underline{\mathbb{P}}(A\cup B) = \mathbb{P}(A\setminus B) + \mathbb{P}(A\cap B) + \mathbb{P}(B\setminus A)$$

$$\mathbb{P}(A) = \mathbb{P}((A\setminus B) \cup (A\cap B))$$

$$= \mathbb{P}(A \setminus B) + \mathbb{P}(A \cap B)$$

Similarly

$$\mathbb{P}(B) = \mathbb{P}(B \setminus A) + \mathbb{P}(A \cap B)$$

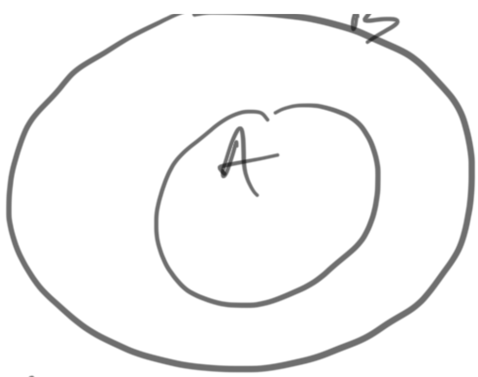
From last three statements we conclude

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$$

Lemma: Let $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, $A, B \in \mathcal{F}$ and $A \subseteq B$. Then, $\mathbb{P}(A) \leq \mathbb{P}(B)$

Proof:

Proof: Here



$B = \underbrace{A \cup (B \setminus A)}$

Using 3rd axiom

$$\mathbb{P}(B) = \mathbb{P}(A) + \mathbb{P}(B \setminus A)$$

Using first axiom

$$\mathbb{P}(B \setminus A) \geq 0$$

So, $\mathbb{P}(B) \geq \mathbb{P}(A)$

Lemma: For any $A, B \in \mathcal{F}$

$$\mathbb{P}(A \cup B) \leq \mathbb{P}(A) + \mathbb{P}(B)$$

Lemma: For any $A_1, A_2, \dots, A_n \in \mathcal{F}$

$$\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} \mathbb{P}(A_i)$$

Lemma: For any $\{A_i\}_{i=1}^{\infty} \subseteq \mathcal{F}$

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mathbb{P}(A_i)$$
