Feb 18

Convergence in Calculus

$\{a_n\}$ is a seq.

$a_n \longrightarrow a$ as $n \to \infty$

if $\underbrace{|a_n - a| \longrightarrow 0}$

Convergence in Prob.

$X_n \xrightarrow{P} X$

if $\underbrace{\mathbb{P}\left(|X_n - X| > \varepsilon \right)} \longrightarrow 0$

or $\mathbb{P}\left(|X_n - X| < \varepsilon \right) \longrightarrow 1$

Convergence in distribution

or

Convergence in law

or

weak convergence

$X_n \xrightarrow{d} X$

$\downarrow$

CDF of $X_n \longrightarrow$ CDF of X

$F_n(x) \longrightarrow F(x)$

for all continuity point x of F.

Eg: let X be a RV

... X_n ...

with CDF

$$\underline{F_n(x)} = \begin{cases} 0 & \text{if } x < n \\ 1 & \text{if } x \geq n \end{cases}$$

i.e. X_n is degenerate at n.

$$\lim_{n \to \infty} \underline{F_n(x)} = \underbrace{0}_{\text{is not a CDF}}$$

<u>Remark (not for exam)</u>

<u>1</u> For defining convergence in probability we need $\{X_n\}_{n}$ and X are

defined on the same probability space.

(ii) For convergence in distribution the RVs may be defined on different probability spaces.

Eg: $X_n \xrightarrow{d} X$

but $X_n \not\xrightarrow{p} X$

in Topic 10 notes.

Convergence in distribution and CLT

iid

$X_1, \ldots, X_n$

with mean μ and variance σ^2.

Define $\bar{X}_n = \dfrac{X_1 + \ldots + X_n}{n}$

then.

$$\frac{\bar{X}_n - \mu}{\sqrt{\operatorname{var}(\bar{X}_n)}} \xrightarrow{d} N(0,1)$$

$$\sqrt{n}\left(\frac{\bar{X}_n - \mu}{\sigma}\right) \xrightarrow{d} \underline{N(0,1)}$$

2 call from WLLN

$$\bar{X}_n \xrightarrow{p} \mu$$

$$\Rightarrow \underbrace{\frac{\bar{X}_n - \mu}{\sigma}} \xrightarrow{p} \underbrace{0}$$

~~in~~ observe

$$n^{1/3}\left(\frac{\bar{X}_n - \mu}{\sigma}\right) \xrightarrow{p} 0$$

$$n^{0.5+\varepsilon}$$

Theorem:

Let $X_1, \ldots, X_n$ are IID Bernoulli(p) RVs.

Then

$$X_1 + \cdots + X_n \sim \text{Binomial}(n, p)$$

Proof: MGF

Theorem: Let $X_n \sim \text{Binomial}(n,p)$

Then,

$$\frac{X_n - np}{\sqrt{np(1-p)}} \xrightarrow{d} N(0,1)$$

Proof: From previous theorem

$$X_n \overset{d}{=} Y_1 + Y_2 + \cdots + Y_n$$

where $Y_i \overset{IID}{\sim} \text{Bernoulli}(p)$

Use CLT.

Theorem: let $X_n \sim \text{Poisson}(n)$.

Then $\dfrac{X_n - n}{\sqrt{n}} \xrightarrow{d} N(0,1)$

Theorem: let $Y_1, Y_2, \dots, Y_n \overset{IID}{\sim} \text{Poisson}(a_i)$

$a_i > 0$

Then,

$Y_1 + \dots + Y_n$

$\sim \text{Poisson}(a_1 + \dots + a_n)$

if $a_i = 1$

then $Y_1 + \dots + Y_n \sim \text{Poisson}(n)$

Proof:

MGF