Feb 11

Function of RV

Methods

(1) Base method:

Using the CDF

$\boxed{eg}$ Let three unbiased coins are tossed and

X = number of heads

$Y = g(X) = |X-2|$

X	0	1	2	3
$g(X)$	2	1	0	1

The CDF of $Y = g(X)$

F_Y is given by

$$F_Y(y) = \mathbb{P}(Y \leq y)$$

let $\underline{y < 0}$

$$F_Y(y) = \mathbb{P}(Y \leq y) = 0$$

let $0 \leq \underline{y < 1}$

$$\underline{F_Y(y)} = \mathbb{P}(Y \leq y)$$

$$= \mathbb{P}(|X-2| \leq y)$$

$$= \mathbb{P}(X = 2)$$

$$= \frac{3}{8}$$

Continuous case

$$\mathbb{P}(g(X) \leq y)$$

$$= \mathbb{P}(\{\cancel{X} : g(\cancel{X}) \leq y\})$$

$$= \int_{\{x : g(x) \leq y\}} f_X(x)\, dx$$

$$y = g(x)$$

$$\stackrel{?}{=} \int_{\{x : g(x) \leq y\}} \boxed{f_X(x) \cdot \left|\frac{dx}{dy}\right|}\, dy$$

$$\downarrow$$

PDF of Y

~~Result~~:

Let $X_1, X_2, \ldots, X_n$ are IID RVs with distribution Uniform $(0, \theta)$, $\theta > 0$.

Define

$$Y_n = \max\{X_1, \ldots, X_n\}$$

we are interested in distribution of Y_n.

For X_1, the CDF

$$F_X(x) = \mathbb{P}(X \leq x)$$

$$= \begin{cases} 0 & \text{if } x < 0 \\ \frac{x}{\theta} & \text{if } \underline{x \in [0, \theta)} \end{cases}$$

$1 \quad \text{if } x \geq \theta$

let $F_{Y_n}(\cdot)$ be the CDF of Y_n. Then

$$F_{Y_n}(y) = \mathbb{P}(Y_n \leq y)$$

$$= \mathbb{P}(\max(X_1, \ldots X_n) \leq y)$$

$$= \mathbb{P}(X_1 \leq y, \ldots, X_n \leq y)$$

indep.

$$= \mathbb{P}(X_1 \leq y) \cdots \mathbb{P}(X_n \leq y)$$

identically

$$= [\mathbb{P}(X_1 \leq y)]^n$$

$$= \left(\frac{y}{\theta}\right)^n, \quad y \in (0, \theta)$$

(θ)

$$\Rightarrow \mathbb{P}(\underline{Y_n} \le y)$$
$$= \left(\frac{y}{\theta}\right)^n, \; y \in (0,\theta)$$

Taking $n \to \infty$

$$= \begin{cases} 0 & \text{if } y < \theta \\ 1 & \text{if } y \ge \theta \end{cases}$$

<u>Theorem</u>: Let $X_1, X_2, \ldots, X_n$ are IID RVs with distribution $N(\mu, \sigma^2)$.

Define $\bar{X}_n = \dfrac{X_1 + \ldots + X_n}{n}$.

Then $\bar{X}_n \sim N(\mu, \sigma/n)$

Hint :-

$$M_{\frac{X_1+\dots+X_n}{n}}(t)$$

$$= \mathbb{E}\left(e^{\frac{X_1+\dots+X_n}{n}t}\right)$$

$$= \mathbb{E}\left(e^{\frac{X_1 t}{n}+\frac{X_2 t}{n}+\dots+\frac{X_n t}{n}}\right)$$

$$= \mathbb{E}\left(e^{\frac{X_1 t}{n}}\right)\dots\mathbb{E}\left(e^{\frac{X_n t}{n}}\right)$$

$$= \underbrace{M_{X_1}\left(\frac{t}{n}\right)}\dots\underbrace{M_{X_n}\left(\frac{t}{n}\right)}$$

Theorem: ~~If X_1 and X_2~~

~~are ind~~

Let $X_1 \sim N(\mu_1, \sigma_1^2)$

and $X_2 \sim N(\mu_2, \sigma_2^2)$

are independent.

Then, for $a_1, a_2 \in \mathbb{R}$,

$$a_1 X_1 + a_2 X_2 \sim N(a_1 \mu_1 + a_2 \mu_2, \; a_1^2 \sigma_1^2 + a_2^2 \sigma_2^2)$$

Markov

Chebychev's

next class