MTL108: Solution to Problem Set-7

Prepared by TAs

IIT delhi

1. **Problem 1**

   In a circuit, the voltages across two resistors are independent random variables, X and Y. The MGF of X is
   $$M_X(t) = (1 - 2t)^{-1}$$
   and the MGF of Y is
   $$M_Y(t) = (1 - 4t)^{-1}.$$

   (a) What are the distributions of X and Y?

   (b) Find the MGF of the total voltage $Z = X + Y$.

   (c) Use the MGF of Z to find $E[Z]$ and $\text{Var}(Z)$.

   (d) What is the distribution of Z?

   (e) Obtain mean, variance, skewness, and kurtosis of Z.

   Solution

   (a) Distributions of X and Y

   The MGF of an exponential distribution with rate λ is
   $$M(t) = \frac{\lambda}{\lambda - t} = (1 - \tfrac{t}{\lambda})^{-1}.$$
   Comparing:
   $$M_X(t) = (1 - 2t)^{-1} \Rightarrow X \sim \text{Exp}\left(\tfrac{1}{2}\right),$$
   $$M_Y(t) = (1 - 4t)^{-1} \Rightarrow Y \sim \text{Exp}\left(\tfrac{1}{4}\right).$$

   (b) MGF of $Z = X + Y$

   Since X and Y are independent,
   $$M_Z(t) = M_X(t)M_Y(t) = (1 - 2t)^{-1}(1 - 4t)^{-1}.$$

   (c) Mean and Variance of Z

   For exponential distribution:
   $$E[X] = 2, \quad \text{Var}(X) = 4,$$
   $$E[Y] = 4, \quad \text{Var}(Y) = 16.$$

Thus,

$$E[Z] = E[X] + E[Y] = 2 + 4 = 6,$$
$$\text{Var}(Z) = \text{Var}(X) + \text{Var}(Y) = 4 + 16 = 20.$$

(d) Distribution of Z

Since X and Y are independent exponentials with different rates, Z follows a **hypoexponential distribution**.

Its PDF is:

$$f_Z(z) = \frac{1}{4-2}\left(e^{-z/4} - e^{-z/2}\right), \quad z > 0.$$

Note: If the rates are the same, then $X + Y$ follows a gamma distribution.

(e) Skewness and Kurtosis

For $X \sim \text{Exp}(\lambda)$, we have:

$$\mathbb{E}[X] = \frac{1}{\lambda}, \quad \text{Var}(X) = \frac{1}{\lambda^2}$$
$$\mu_3 = \frac{2}{\lambda^3}, \quad \mu_4 = \frac{9}{\lambda^4}$$

For $X \sim \text{Exp}(2)$:

$$\mu_3(X) = 2(2)^3 = 16, \quad \mu_4(X) = 9(2)^4 = 144$$

For $Y \sim \text{Exp}(4)$:

$$\mu_3(Y) = 2(4)^3 = 128, \quad \mu_4(Y) = 9(4)^4 = 2304$$

Since X and Y are independent:

$$\mu_3(Z) = \mu_3(X) + \mu_3(Y) = 16 + 128 = 144$$
$$\mu_4(Z) = \mu_4(X) + \mu_4(Y) = 144 + 2304 = 2448$$

$$\text{Skewness} = \frac{\mu_3(Z)}{(\text{Var}(Z))^{3/2}} = \frac{144}{20^{3/2}}$$

$$\text{Kurtosis} = \frac{\mu_4(Z)}{(\text{Var}(Z))^2} = \frac{2448}{400} = 6.12$$

2. **Problem 2**

Obtain expressions for median, mode, skewness, and kurtosis for the following distributions.

Bernoulli, Binomial, Poisson, Geometric, Negative Binomial, Uniform discrete, Uniform continuous, Exponential, Normal.

(a) **Bernoulli Distribution:** Let $X \sim \text{Bernoulli}(p)$, where

$$P(X = 1) = p, \quad P(X = 0) = 1 - p.$$

Mode

The mode is the value with the highest probability.

- If $p > \frac{1}{2}$, then
$$P(X = 1) > P(X = 0) \implies \text{Mode} = 1.$$
- If $p < \frac{1}{2}$, then
$$P(X = 0) > P(X = 1) \implies \text{Mode} = 0.$$
- If $p = \frac{1}{2}$, then
$$P(X = 0) = P(X = 1),$$
so both 0 and 1 are modes (bimodal distribution).

Median

A median m satisfies
$$P(X \le m) \ge \frac{1}{2} \quad \text{and} \quad P(X \ge m) \ge \frac{1}{2}.$$

Case 1: $p < \frac{1}{2}$
$$P(X \le 0) = P(X = 0) = 1 - p > \frac{1}{2}.$$
Hence,
$$\text{Median} = 0.$$

Case 2: $p > \frac{1}{2}$
$$P(X \le 0) = 1 - p < \frac{1}{2}, \quad P(X \le 1) = 1.$$
Hence,
$$\text{Median} = 1.$$

Case 3: $p = \frac{1}{2}$
$$P(X \le 0) = \frac{1}{2}, \quad P(X \ge 0) = 1,$$
and
$$P(X \le 1) = 1, \quad P(X \ge 1) = \frac{1}{2}.$$
Also,
$$P(X \le 0.5) = \frac{1}{2}, \quad P(X \ge 0.5) = \frac{1}{2}.$$
Thus, any value in $[0, 1]$ is a median.

Mean and Variance

$$\mu = E[X] = p$$
Since $X^2 = X$,
$$E[X^2] = p$$
$$\text{Var}(X) = E[X^2] - \mu^2 = p - p^2 = p(1 - p)$$
Let $\sigma^2 = p(1 - p)$.

Third Central Moment

$$\mu_3 = E[(X-p)^3]$$

$$\mu_3 = p(1-p)^3 + (1-p)(-p)^3$$

$$= p(1-3p+3p^2-p^3) - (1-p)p^3$$

$$= p - 3p^2 + 3p^3 - p^4 - p^3 + p^4$$

$$= p - 3p^2 + 2p^3$$

$$= p(1-3p+2p^2) = p(1-p)(1-2p)$$

Skewness

$$\text{Skewness} = \frac{\mu_3}{\sigma^3}$$

$$= \frac{p(1-p)(1-2p)}{[p(1-p)]^{3/2}} = \frac{1-2p}{\sqrt{p(1-p)}}$$

Fourth Central Moment

$$\mu_4 = E[(X-p)^4]$$

$$\mu_4 = p(1-p)^4 + (1-p)p^4$$

$$= p(1-4p+6p^2-4p^3+p^4) + (1-p)p^4$$

$$= p - 4p^2 + 6p^3 - 4p^4 + p^5 + p^4 - p^5$$

$$= p - 4p^2 + 6p^3 - 3p^4$$

$$= p(1-4p+6p^2-3p^3)$$

Kurtosis

$$\text{Kurtosis} = \frac{\mu_4}{\sigma^4}$$

$$= \frac{p(1-p)(1-3p+3p^2)}{[p(1-p)]^2} = \frac{1-3p+3p^2}{p(1-p)}$$

(b) **Binomial Distribution:**
Let $X \sim \text{Binomial}(n,p)$, where

$$P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, 1, \ldots, n.$$

Mode

To find the mode, consider the ratio:

$$\frac{P(X=k+1)}{P(X=k)} = \frac{\binom{n}{k+1}p^{k+1}(1-p)^{n-k-1}}{\binom{n}{k}p^k(1-p)^{n-k}}$$

Using

$$\frac{\binom{n}{k+1}}{\binom{n}{k}} = \frac{n-k}{k+1},$$

we get:

$$\frac{P(X=k+1)}{P(X=k)} = \frac{n-k}{k+1} \cdot \frac{p}{1-p}.$$

The pmf increases as long as this ratio is ≥ 1:

$$\frac{n-k}{k+1} \cdot \frac{p}{1-p} \geq 1.$$

Solving:

$$(n-k)p \geq (k+1)(1-p)$$

$$np - kp \geq k + 1 - kp - p$$

$$np \geq k + 1 - p$$

$$k \leq (n+1)p - 1.$$

Thus, the largest integer satisfying this is:

$$\text{Mode} = \lfloor (n+1)p \rfloor$$

provided $(n+1)p$ is not an integer.
If $(n+1)p$ is an integer, then there are two modes:

$$(n+1)p - 1 \quad \text{and} \quad (n+1)p$$

as $P((n+1)p-1) = P((n+1)p)$ (check yourself).

Median

The median m satisfies:

$$P(X \leq m) \geq \frac{1}{2}, \quad P(X \geq m) \geq \frac{1}{2}.$$

Unlike the mode, the median does not have a simple closed form. However, it satisfies the inequality:

$$\lfloor (n+1)p \rfloor - 1 \leq m \leq \lfloor (n+1)p \rfloor,$$

as the probability increases up to the mode and then decreases.
Let $X_i \sim \text{Bernoulli}(n,p)$, independent, then

$$X = \sum_{i=1}^{n} X_i$$

Mean and Variance

$$E[X] = \sum_{i=1}^{n} E[X_i] = np$$

$$\mathrm{Var}(X) = \sum_{i=1}^{n} \mathrm{Var}(X_i) = np(1-p)$$

Let $\mu = np$ and $\sigma^2 = np(1-p)$.

Third Central Moment

For a Bernoulli random variable,

$$\mu_3(X_i) = p(1-p)(1-2p).$$

Since the X_i are independent,

$$\mu_3(X) = \sum_{i=1}^{n} \mu_3(X_i) = np(1-p)(1-2p).$$

Skewness

$$\gamma_1 = \frac{\mu_3(X)}{\sigma^3} = \frac{np(1-p)(1-2p)}{[np(1-p)]^{3/2}} = \frac{1-2p}{\sqrt{np(1-p)}}.$$

Let $X \sim \text{Binomial}(n, p)$. Then

$$X = \sum_{i=1}^{n} X_i, \quad X_i \sim \text{Bernoulli}(p), \text{ independent.}$$

Let

$$\mu = E[X] = np, \quad \sigma^2 = \mathrm{Var}(X) = np(1-p).$$

Fourth Central Moment

$$\mu_4 = E[(X-\mu)^4] = E\left[\left(\sum_{i=1}^{n}(X_i - p)\right)^4\right].$$

Let $Y_i = X_i - p$. Then $E[Y_i] = 0$.
Using expansion:

$$\left(\sum_{i=1}^{n} Y_i\right)^4 = \sum_{i=1}^{n} Y_i^4 + 6\sum_{i<j} Y_i^2 Y_j^2 + (\text{terms with odd powers}).$$

Taking expectation and using independence:

$$\mu_4 = \sum_{i=1}^{n} E[Y_i^4] + 6\sum_{i<j} E[Y_i^2]E[Y_j^2].$$

(expectation of odd terms becomes zero as $E[Y_i] = 0$)
We will compute each term separately. For the first term,

$$\sum_{i=1}^{n} E[Y_i^4] = nE[(X_1 - p)^4].$$

For Bernoulli:

$$E[(X_1 - p)^4] = p(1-p)^4 + (1-p)p^4 = p(1-p)(1 - 3p + 3p^2).$$

Thus,

$$\sum_{i=1}^{n} E[Y_i^4] = np(1-p)(1 - 3p + 3p^2).$$

For the second term,

$$E[Y_i^2] = \mathrm{Var}(X_i) = p(1-p).$$

Hence,

$$6\sum_{i<j} E[Y_i^2]E[Y_j^2] = 6 \cdot \frac{n(n-1)}{2}[p(1-p)]^2 = 3n(n-1)[p(1-p)]^2.$$

Combining both terms, we get

$$\mu_4 = np(1-p)(1 - 3p + 3p^2) + 3n(n-1)[p(1-p)]^2.$$

Kurtosis

$$\beta_2 = \frac{\mu_4}{\sigma^4}, \quad \sigma^2 = np(1-p).$$

$$\sigma^4 = [np(1-p)]^2.$$

Thus,

$$\beta_2 = \frac{np(1-p)(1 - 3p + 3p^2)}{[np(1-p)]^2} + \frac{3n(n-1)[p(1-p)]^2}{[np(1-p)]^2}.$$

Simplifying,

$$\beta_2 = \frac{1 - 3p + 3p^2}{np(1-p)} + 3\frac{n-1}{n}.$$

Rewriting,

$$\beta_2 = 3 + \frac{1 - 6p(1-p)}{np(1-p)}.$$

(c) **Poisson Distribution**
Let $X \sim \mathrm{Poisson}(\lambda)$ with pmf

$$P(X = k) = \frac{e^{-\lambda}\lambda^k}{k!}, \quad k = 0, 1, 2, \ldots$$

Mode

To find the mode, consider the ratio:

$$\frac{P(X=k+1)}{P(X=k)} = \frac{\frac{e^{-\lambda}\lambda^{k+1}}{(k+1)!}}{\frac{e^{-\lambda}\lambda^{k}}{k!}}$$

$$= \frac{\lambda^{k+1}}{(k+1)!} \cdot \frac{k!}{\lambda^k} = \frac{\lambda}{k+1}.$$

The pmf increases as long as:

$$\frac{\lambda}{k+1} \geq 1 \quad \Rightarrow \quad \lambda \geq k+1 \quad \Rightarrow \quad k \leq \lambda - 1.$$

Thus, the largest integer satisfying this is:

$$k = \lfloor \lambda \rfloor - 1.$$

Hence, the mode occurs at:

$$\text{Mode} = \lfloor \lambda \rfloor.$$

Special Case

If $\lambda = m \in \mathbb{Z}$, then at $k = m-1$,

$$\frac{P(m)}{P(m-1)} = \frac{\lambda}{m} = 1 \quad \Rightarrow \quad P(m) = P(m-1).$$

Thus, there are two modes:

$$\boxed{m-1 \text{ and } m}.$$

Median

The median m satisfies:

$$P(X \leq m) \geq \frac{1}{2}, \quad P(X \geq m) \geq \frac{1}{2}.$$

Unlike the mode, there is no closed-form expression for the median of a Poisson distribution.
However, it satisfies the bounds:

$$\lambda - \ln 2 \leq m \leq \lambda + \frac{1}{3}.$$

A median m satisfies:

$$P(X \leq m) \geq \frac{1}{2}, \quad P(X \geq m) \geq \frac{1}{2}.$$

For Lower Bound: $m \geq \lambda - \ln 2$
We bound $P(X \leq a)$ using Markov's inequality.
We have:

$$P(X \leq a) = P(e^{-X} \geq e^{-a}).$$

By Markov's inequality,

$$P(e^{-X} \geq e^{-a}) \leq \frac{E[e^{-X}]}{e^{-a}}.$$

Now compute $E[e^{-X}]$:

$$E[e^{-X}] = \sum_{k=0}^{\infty} e^{-k} \frac{e^{-\lambda}\lambda^k}{k!} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda e^{-1})^k}{k!}.$$

Using the exponential series,

$$\sum_{k=0}^{\infty} \frac{(\lambda e^{-1})^k}{k!} = e^{\lambda e^{-1}},$$

so,

$$E[e^{-X}] = e^{-\lambda(1-e^{-1})}.$$

Thus,

$$P(X \leq a) \leq \frac{e^{-\lambda(1-e^{-1})}}{e^{-a}} = e^{-\lambda(1-e^{-1})+a}.$$

To ensure $P(X \leq a) \leq \frac{1}{2}$, we require:

$$e^{-\lambda(1-e^{-1})+a} \leq \frac{1}{2}.$$

Taking the logarithm,

$$-\lambda(1-e^{-1}) + a \leq -\ln 2,$$

$$a \leq \lambda(1-e^{-1}) - \ln 2.$$

This yields the approximate lower bound:

$$m \geq \lambda - \ln 2.$$

For upper bound: $m \leq \lambda + \frac{1}{3}$
Using a Chernoff-type (see last page) bound for the Poisson distribution:

$$P(X \geq \lambda + t) \leq \exp\left(-\frac{t^2}{2(\lambda + t/3)}\right).$$

Choose $t = \frac{1}{3}$. Then:

$$P\left(X \geq \lambda + \frac{1}{3}\right) \leq \frac{1}{2}.$$

Thus, the median must satisfy:

$$m \leq \lambda + \frac{1}{3}.$$

Combining both bounds:

$$\lambda - \ln 2 \leq m \leq \lambda + \frac{1}{3}.$$

Mean and Variance

$$E[X] = \lambda, \quad \text{Var}(X) = \lambda.$$

We have,

$$E[X(X-1)] = \lambda^2, \quad E[X(X-1)(X-2)] = \lambda^3,$$
$$E[X(X-1)(X-2)(X-3)] = \lambda^4.$$

$$E[X^2] = \lambda^2 + \lambda,$$

$$E[X^3] = \lambda^3 + 3\lambda^2 + \lambda,$$

$$E[X^4] = \lambda^4 + 6\lambda^3 + 7\lambda^2 + \lambda.$$

Third Central Moment

$$\mu_3 = E[X^3] - 3\lambda E[X^2] + 3\lambda^2 E[X] - \lambda^3 = \lambda.$$

Fourth Central Moment

$$\mu_4 = E[X^4] - 4\lambda E[X^3] + 6\lambda^2 E[X^2] - 4\lambda^3 E[X] + \lambda^4 = \lambda + 3\lambda^2.$$

Skewness

$$\gamma_1 = \frac{\mu_3}{\sigma^3} = \frac{\lambda}{\lambda^{3/2}} = \frac{1}{\sqrt{\lambda}}.$$

Kurtosis

$$\beta_2 = \frac{\mu_4}{\sigma^4} = \frac{\lambda + 3\lambda^2}{\lambda^2} = 3 + \frac{1}{\lambda}.$$

(d) **Geometric distribution**
Let $X \sim \text{Geometric}(p)$ with pmf

$$P(X = k) = (1-p)^{k-1}p, \quad k = 1, 2, \ldots$$

Mode

Consider the ratio:
$$\frac{P(X = k+1)}{P(X = k)} = \frac{(1-p)^k p}{(1-p)^{k-1}p} = 1 - p.$$
Since $0 < p \leq 1$, we have:
$$1 - p < 1.$$

Thus,
$$P(X = k+1) < P(X = k),$$
so the pmf is strictly decreasing.
Hence,
$$\text{Mode} = 1.$$

2. Median

The median m satisfies:

$$P(X \leq m) \geq \frac{1}{2}.$$

Now,

$$P(X \leq m) = \sum_{k=1}^{m} (1-p)^{k-1} p = p \sum_{k=0}^{m-1} (1-p)^k.$$

Using geometric series:

$$\sum_{k=0}^{m-1} (1-p)^k = \frac{1-(1-p)^m}{p}.$$

Thus,

$$P(X \leq m) = 1-(1-p)^m.$$

So,

$$1-(1-p)^m \geq \frac{1}{2} \Rightarrow (1-p)^m \leq \frac{1}{2}.$$

Taking logarithm:

$$m \ln(1-p) \leq \ln \frac{1}{2}.$$

Since $\ln(1-p) < 0$, inequality reverses:

$$m \geq \frac{\ln(1/2)}{\ln(1-p)}.$$

Hence,

$$m = \left\lceil \frac{\ln(1/2)}{\ln(1-p)} \right\rceil.$$

Now,
We first compute $E[X]$:

$$E[X] = \sum_{k=1}^{\infty} k(1-p)^{k-1} p.$$

Using the identity:

$$\sum_{k=1}^{\infty} k r^{k-1} = \frac{1}{(1-r)^2}, \quad |r| < 1,$$

with $r = 1-p$, we get:

$$E[X] = p \cdot \frac{1}{p^2} = \frac{1}{p}.$$

Now,

$$E[X^2] = \sum_{k=1}^{\infty} k^2 (1-p)^{k-1} p.$$

Using:

$$\sum_{k=1}^{\infty} k^2 r^{k-1} = \frac{1+r}{(1-r)^3},$$

we get:

$$E[X^2] = p \cdot \frac{1 + (1-p)}{p^3} = \frac{2-p}{p^2}.$$

Thus,

$$\text{Var}(X) = E[X^2] - (E[X])^2 = \frac{2-p}{p^2} - \frac{1}{p^2} = \frac{1-p}{p^2}.$$

Third Central Moment

Using:

$$\mu_3 = E[X^3] - 3\mu E[X^2] + 3\mu^2 E[X] - \mu^3,$$

we first compute:

$$\sum_{k=1}^{\infty} k^3 r^{k-1} = \frac{1 + 4r + r^2}{(1-r)^4}.$$

Thus,

$$E[X^3] = p \cdot \frac{1 + 4(1-p) + (1-p)^2}{p^4}.$$

Simplifying,

$$E[X^3] = \frac{6 - 6p + p^2}{p^3}.$$

Now substitute:

$$\mu = \frac{1}{p}, \quad E[X^2] = \frac{2-p}{p^2}.$$

After simplification:

$$\mu_3 = \frac{(2-p)(1-p)}{p^3}.$$

Skewness

$$\gamma_1 = \frac{\mu_3}{\sigma^3}, \quad \sigma^2 = \frac{1-p}{p^2}.$$

$$\sigma^3 = \frac{(1-p)^{3/2}}{p^3}.$$

Thus,

$$\gamma_1 = \frac{(2-p)(1-p)}{p^3} \cdot \frac{p^3}{(1-p)^{3/2}} = \frac{2-p}{\sqrt{1-p}}.$$

$$\gamma_1 = \frac{2-p}{\sqrt{1-p}}.$$

Fourth Central Moment

Using expansion,

$$E[X^4] = p \cdot \frac{1 + 11r + 11r^2 + r^3}{(1-r)^5}, \quad r = 1 - p.$$

After simplification and substituting into:

$$\mu_4 = E[X^4] - 4\mu E[X^3] + 6\mu^2 E[X^2] - 4\mu^3 E[X] + \mu^4,$$

we obtain:

$$\mu_4 = \frac{(1-p)(6 - 6p + p^2)}{p^4}.$$

Kurtosis

$$\beta_2 = \frac{\mu_4}{\sigma^4}, \quad \sigma^4 = \frac{(1-p)^2}{p^4}.$$

$$\beta_2 = \frac{(1-p)(6-6p+p^2)}{p^4} \cdot \frac{p^4}{(1-p)^2} = \frac{6 - 6p + p^2}{1-p}.$$

$$= 6 + \frac{p^2}{1-p}.$$

$$\beta_2 = 6 + \frac{p^2}{1-p}.$$

(e) **Negative binomial distribution**
Let $X \sim \text{NegBin}(r, p)$ (number of trials until r-th success) with pmf

$$P(X = k) = \binom{k-1}{r-1} p^r (1-p)^{k-r}, \quad k = r, r+1, \ldots$$

Mode

Consider the ratio:

$$\frac{P(X = k+1)}{P(X = k)} = \frac{\binom{k}{r-1} p^r (1-p)^{k+1-r}}{\binom{k-1}{r-1} p^r (1-p)^{k-r}}.$$

Using:

$$\frac{\binom{k}{r-1}}{\binom{k-1}{r-1}} = \frac{k}{k - (r-1)},$$

we get:

$$\frac{P(X = k+1)}{P(X = k)} = \frac{k}{k - r + 1}(1-p).$$

For increasing pmf:

$$\frac{k}{k-r+1}(1-p) \geq 1.$$

$$k(1-p) \geq k - r + 1 \Rightarrow k - kp \geq k - r + 1 \Rightarrow -kp \geq -r + 1 \Rightarrow kp \leq r - 1.$$

$$k \le \frac{r-1}{p}.$$

Thus, mode occurs at:

$$\text{Mode} = \left\lfloor \frac{r-1}{p} \right\rfloor + r.$$

Median

Similar to binomial, it also doesn't have a closed-form expression.

Let $M_X(t)$ be the MGF:

$$M_X(t) = \left(\frac{pe^t}{1-(1-p)e^t} \right)^r.$$

Let $M(t) = M_X(t)$.

$$M'(t) = rM(t) \left[1 + \frac{(1-p)e^t}{1-(1-p)e^t} \right].$$

At $t = 0$, $M(0) = 1$:

$$E[X] = M'(0) = \frac{r}{p}.$$

Differentiate again:

$$M''(t) = rM'(t) \left[1 + \frac{(1-p)e^t}{1-(1-p)e^t} \right] + rM(t) \frac{(1-p)e^t}{\left(1-(1-p)e^t\right)^2}.$$

At $t = 0$:

$$E[X^2] = M''(0) = \frac{r(r+1)}{p^2}.$$

Thus,

$$\text{Var}(X) = E[X^2] - (E[X])^2 = \frac{r(1-p)}{p^2}.$$

Differentiating again (after simplification):

$$E[X^3] = M^{(3)}(0) = \frac{r(r+1)(r+2)}{p^3}.$$

$$E[X^4] = M^{(4)}(0) = \frac{r(r+1)(r+2)(r+3)}{p^4}.$$

Converting to central moments:
Let $\mu = \frac{r}{p}$.

Third central moment:

$$\mu_3 = E[X^3] - 3\mu E[X^2] + 3\mu^2 E[X] - \mu^3.$$

Substituting:

$$\mu_3 = \frac{r(r+1)(r+2)}{p^3} - 3\frac{r}{p} \cdot \frac{r(r+1)}{p^2} + 3\frac{r^2}{p^2} \cdot \frac{r}{p} - \frac{r^3}{p^3}.$$

Simplifying:

$$\mu_3 = \frac{r(1-p)(2-p)}{p^3}.$$

Skewness

$$\gamma_1 = \frac{\mu_3}{\sigma^3}, \quad \sigma^2 = \frac{r(1-p)}{p^2}.$$

$$\sigma^3 = \frac{r^{3/2}(1-p)^{3/2}}{p^3}.$$

Thus,

$$\gamma_1 = \frac{2-p}{\sqrt{r(1-p)}}.$$

Fourth Central Moment

$$\mu_4 = E[X^4] - 4\mu E[X^3] + 6\mu^2 E[X^2] - 4\mu^3 E[X] + \mu^4.$$

Substitute all values:

$$\mu_4 = \frac{r(r+1)(r+2)(r+3)}{p^4} - 4\frac{r}{p} \cdot \frac{r(r+1)(r+2)}{p^3} + 6\frac{r^2}{p^2} \cdot \frac{r(r+1)}{p^2} - 4\frac{r^3}{p^3} \cdot \frac{r}{p} + \frac{r^4}{p^4}.$$

Simplifying:

$$\mu_4 = \frac{r(1-p)(6-6p+p^2)}{p^4} + 3\frac{r^2(1-p)^2}{p^4}.$$

Kurtosis

$$\beta_2 = \frac{\mu_4}{\sigma^4}, \quad \sigma^4 = \frac{r^2(1-p)^2}{p^4}.$$

Thus,

$$\beta_2 = 3 + \frac{6-6p+p^2}{r(1-p)}.$$

(f) **Uniform discrete distribution**
Let $X \sim \text{Uniform}\{1, 2, \ldots, n\}$:

$$P(X=k) = \frac{1}{n}, \quad k = 1, \ldots, n$$

Mode

All probabilities are equal:

$$\boxed{\text{All values } 1, \ldots, n \text{ are modes}}$$

Median

$$P(X \le m) = \frac{m}{n} \Rightarrow m \ge \frac{n}{2}$$

$$m = \begin{cases} \frac{n+1}{2}, & n \text{ odd} \\ \frac{n}{2}, \frac{n}{2}+1, & n \text{ even} \end{cases}$$

$$E[X] = \frac{1}{n}\sum_{k=1}^{n} k = \frac{n+1}{2} = \mu.$$

Third Central Moment

$$\mu_3 = E[(X-\mu)^3] = \frac{1}{n}\sum_{k=1}^{n}(k-\mu)^3.$$

Since the values $1, 2, \ldots, n$ are symmetric about $\mu = \frac{n+1}{2}$, for each term $(k-\mu)^3$ there exists a corresponding term $(n+1-k-\mu)^3 = -(k-\mu)^3$.
Thus, terms cancel pairwise:

$$\boxed{\mu_3 = 0}.$$

Skewness

$$\gamma_1 = \frac{\mu_3}{\sigma^3} = 0.$$

$$\boxed{\gamma_1 = 0}.$$

Fourth Central Moment

$$\mu_4 = E[(X-\mu)^4] = \frac{1}{n}\sum_{k=1}^{n}(k-\mu)^4.$$

Expand:

$$(k-\mu)^4 = k^4 - 4\mu k^3 + 6\mu^2 k^2 - 4\mu^3 k + \mu^4.$$

Thus,

$$\mu_4 = \frac{1}{n}\sum k^4 - \frac{4\mu}{n}\sum k^3 + \frac{6\mu^2}{n}\sum k^2 - \frac{4\mu^3}{n}\sum k + \mu^4.$$

Using standard sums,

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2},$$

$$\sum k^2 = \frac{n(n+1)(2n+1)}{6},$$

$$\sum k^3 = \left(\frac{n(n+1)}{2}\right)^2,$$

$$\sum k^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}.$$

Substituting $\mu = \frac{n+1}{2}$ all values into μ_4 and simplifying (after algebraic reduction), we get:

$$\boxed{\mu_4 = \frac{(n^2-1)(3n^2-7)}{240}}.$$

$$\sigma^2 = \mathrm{Var}(X) = \frac{n^2-1}{12}.$$

$$\sigma^4 = \frac{(n^2-1)^2}{144}.$$

Kurtosis

$$\beta_2 = \frac{\mu_4}{\sigma^4} = \frac{\frac{(n^2-1)(3n^2-7)}{240}}{\frac{(n^2-1)^2}{144}}.$$

$$= \frac{144(3n^2-7)}{240(n^2-1)}.$$

$$= \boxed{\frac{3(3n^2-7)}{5(n^2-1)}}.$$

(g) **Uniform continuous distribution**
Let $X \sim \text{Uniform}(a, b)$ with pdf:

$$f(x) = \frac{1}{b-a}, \quad a \le x \le b.$$

Mode

Since the density is constant over (a, b):

$$f(x) = \frac{1}{b-a}, \quad \forall x \in (a, b).$$

Thus, all values are equally likely:

$$\text{Every } x \in (a, b) \text{ is a mode.}$$

Median

Median m satisfies:

$$P(X \le m) = \frac{1}{2}.$$

Now,

$$P(X \le m) = \int_a^m \frac{1}{b-a} dx = \frac{m-a}{b-a}.$$

So,

$$\frac{m-a}{b-a} = \frac{1}{2}.$$

$$m - a = \frac{b-a}{2} \Rightarrow m = \frac{a+b}{2}.$$

$$m = \frac{a+b}{2}.$$

$$E[X] = \int_a^b x \frac{1}{b-a} dx = \frac{1}{b-a} \cdot \frac{b^2-a^2}{2}.$$

$$= \frac{a+b}{2}.$$

$$\mu = \frac{a+b}{2}.$$

$$E[X^2] = \int_a^b x^2 \frac{1}{b-a} dx = \frac{1}{b-a} \cdot \frac{b^3 - a^3}{3}.$$

$$= \frac{a^2 + ab + b^2}{3}.$$

$$\text{Var}(X) = E[X^2] - \mu^2 = \frac{a^2 + ab + b^2}{3} - \left(\frac{a+b}{2}\right)^2.$$

$$= \frac{(b-a)^2}{12}.$$

$$\boxed{\sigma^2 = \frac{(b-a)^2}{12}}.$$

Skewness

$$\mu_3 = E[(X-\mu)^3] = \int_a^b (x-\mu)^3 \frac{1}{b-a} dx.$$

Let $y = x - \mu$, then limits become symmetric:

$$y \in \left[-\frac{b-a}{2}, \frac{b-a}{2}\right].$$

Thus,

$$\mu_3 = \frac{1}{b-a} \int_{-(b-a)/2}^{(b-a)/2} y^3 dy.$$

Since y^3 is an odd function:

$$\int_{-c}^{c} y^3 dy = 0.$$

Hence,

$$\boxed{\mu_3 = 0}.$$

$$\boxed{\gamma_1 = 0}.$$

Kurtosis

$$\mu_4 = \frac{1}{b-a} \int_a^b (x-\mu)^4 dx.$$

Using substitution $y = x - \mu$:

$$\mu_4 = \frac{1}{b-a} \int_{-(b-a)/2}^{(b-a)/2} y^4 dy.$$

$$= \frac{2}{b-a} \int_0^{(b-a)/2} y^4 dy.$$

$$= \frac{2}{b-a} \cdot \frac{\left(\frac{b-a}{2}\right)^5}{5}.$$

$$= \frac{(b-a)^4}{80}.$$

Now,

$$\sigma^4 = \left(\frac{(b-a)^2}{12}\right)^2 = \frac{(b-a)^4}{144}.$$

7. Kurtosis

$$\beta_2 = \frac{\mu_4}{\sigma^4} = \frac{\frac{(b-a)^4}{80}}{\frac{(b-a)^4}{144}}.$$

$$= \frac{144}{80} = \frac{9}{5}.$$

$$\boxed{\beta_2 = \frac{9}{5}}.$$

(h) **Exponential Distribution**

Let $X \sim \text{Exponential}(\lambda)$ with pdf:

$$f(x) = \lambda e^{-\lambda x}, \quad x \geq 0.$$

Mode

To find the mode, differentiate $f(x)$:

$$f'(x) = -\lambda^2 e^{-\lambda x} < 0.$$

Thus, $f(x)$ is decreasing on $[0, \infty)$, so maximum occurs at $x = 0$.

$$\text{Mode} = 0.$$

Median

Median m satisfies:

$$P(X \leq m) = \frac{1}{2}.$$

$$\int_0^m \lambda e^{-\lambda x} dx = \frac{1}{2}.$$

$$1 - e^{-\lambda m} = \frac{1}{2}.$$

$$e^{-\lambda m} = \frac{1}{2}.$$

$$m = \frac{\ln 2}{\lambda}.$$

$$\boxed{m = \frac{\ln 2}{\lambda}}.$$

$$E[X] = \int_0^\infty x\lambda e^{-\lambda x} dx.$$

Using integration by parts:

$$E[X] = \frac{1}{\lambda}.$$

$$\mu = \frac{1}{\lambda}.$$

Second Moment

$$E[X^2] = \int_0^\infty x^2 \lambda e^{-\lambda x} dx.$$

Using integration by parts twice:

$$E[X^2] = \frac{2}{\lambda^2}.$$

$$\text{Var}(X) = E[X^2] - \mu^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}.$$

$$\sigma^2 = \frac{1}{\lambda^2}.$$

Third central moment

$$E[X^3] = \int_0^\infty x^3 \lambda e^{-\lambda x} dx = \frac{6}{\lambda^3}.$$

Now,

$$\mu_3 = E[X^3] - 3\mu E[X^2] + 3\mu^2 E[X] - \mu^3.$$

Substitute:

$$\mu_3 = \frac{6}{\lambda^3} - 3\frac{1}{\lambda} \cdot \frac{2}{\lambda^2} + 3\frac{1}{\lambda^2} \cdot \frac{1}{\lambda} - \frac{1}{\lambda^3}.$$

$$= \frac{2}{\lambda^3}.$$

Skewness

$$\gamma_1 = \frac{\mu_3}{\sigma^3}.$$

$$\sigma^3 = \frac{1}{\lambda^3}.$$

$$\gamma_1 = \frac{2/\lambda^3}{1/\lambda^3} = 2.$$

$$\gamma_1 = 2.$$

Fourth Central Moment

$$E[X^4] = \int_0^\infty x^4 \lambda e^{-\lambda x} dx = \frac{24}{\lambda^4}.$$

$$\mu_4 = E[X^4] - 4\mu E[X^3] + 6\mu^2 E[X^2] - 4\mu^3 E[X] + \mu^4.$$

Substitute:

$$\mu_4 = \frac{24}{\lambda^4} - 4\frac{1}{\lambda} \cdot \frac{6}{\lambda^3} + 6\frac{1}{\lambda^2} \cdot \frac{2}{\lambda^2} - 4\frac{1}{\lambda^3} \cdot \frac{1}{\lambda} + \frac{1}{\lambda^4}.$$

$$= \frac{9}{\lambda^4}.$$

Kurtosis

$$\beta_2 = \frac{\mu_4}{\sigma^4}.$$

$$\sigma^4 = \frac{1}{\lambda^4}.$$

$$\beta_2 = \frac{9/\lambda^4}{1/\lambda^4} = 9.$$

$$\beta_2 = 9.$$

(i) **Normal distribution**
Let $X \sim N(\mu, \sigma^2)$ with pdf:

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right).$$

Mode

Differentiate $f(x)$:

$$f'(x) = f(x)\left(-\frac{x-\mu}{\sigma^2}\right).$$

Set $f'(x) = 0$:

$$x - \mu = 0 \Rightarrow x = \mu.$$

Thus,

$$\text{Mode} = \mu.$$

Median

Median m satisfies:

$$P(X \le m) = \frac{1}{2}.$$

Since the normal distribution is symmetric about μ:

$$P(X \le \mu) = \frac{1}{2}.$$

Thus,

$$m = \mu.$$

By definition:

$$E[X] = \mu.$$

$$\text{Mean} = \mu.$$

$$\text{Var}(X) = \sigma^2.$$

Third Central Moment

$$\mu_3 = E[(X-\mu)^3].$$

Let $Y = X - \mu \sim N(0, \sigma^2)$.

$$\mu_3 = E[Y^3] = \int_{-\infty}^{\infty} y^3 \frac{1}{\sqrt{2\pi}\sigma} e^{-y^2/(2\sigma^2)} dy.$$

Since y^3 is an odd function and the density is symmetric:

$$\mu_3 = 0.$$

Skewness

$$\gamma_1 = \frac{\mu_3}{\sigma^3} = 0.$$

$$\gamma_1 = 0.$$

Fourth Central Moment

$$\mu_4 = E[(X-\mu)^4].$$

Let $Y = X - \mu$:

$$\mu_4 = E[Y^4].$$

Using standard integral (or integration by parts):

$$E[Y^4] = 3\sigma^4.$$

$$\mu_4 = 3\sigma^4.$$

Kurtosis

$$\beta_2 = \frac{\mu_4}{\sigma^4} = \frac{3\sigma^4}{\sigma^4} = 3.$$

$$\beta_2 = 3.$$

Chernoff Bound

Let X be a random variable. For any $t > 0$,

$$P(X \geq a) = P(e^{tX} \geq e^{ta}).$$

Using Markov's inequality,

$$P(e^{tX} \geq e^{ta}) \leq \frac{E[e^{tX}]}{e^{ta}}.$$

Thus,

$$P(X \geq a) \leq \inf_{t>0} \frac{E[e^{tX}]}{e^{ta}}$$

This is called a **Chernoff bound**.

Application to Poisson Distribution

Let $X \sim \text{Poisson}(\lambda)$. Its moment generating function is:

$$E[e^{tX}] = \exp\left(\lambda(e^t - 1)\right).$$

Hence,

$$P(X \geq a) \leq \inf_{t>0} \exp\left(\lambda(e^t - 1) - ta\right).$$

Minimize:

$$\lambda(e^t - 1) - ta.$$

Differentiating:

$$\lambda e^t - a = 0 \quad \Rightarrow \quad e^t = \frac{a}{\lambda}.$$

Thus,

$$t = \ln\left(\frac{a}{\lambda}\right).$$

Substituting,

$$P(X \geq a) \leq \exp\left(a - \lambda - a\ln\frac{a}{\lambda}\right).$$

$$\boxed{P(X \geq a) \leq \exp\left(-\lambda + a - a\ln\frac{a}{\lambda}\right)}$$

Use in Bounding the Median

Let m be the median. Then:

$$P(X \geq m) \leq \frac{1}{2}.$$

Set $a = \lambda + t$. Using the Chernoff bound:

$$P(X \geq \lambda + t) \leq \exp\left(-\lambda + (\lambda + t) - (\lambda + t)\ln\frac{\lambda + t}{\lambda}\right).$$

For small t, this can be simplified to a standard bound:

$$P(X \geq \lambda + t) \leq \exp\left(-\frac{t^2}{2(\lambda + t/3)}\right).$$

Choosing $t = \frac{1}{3}$, the right-hand side becomes less than $\frac{1}{2}$, implying:

$$P\left(X \geq \lambda + \frac{1}{3}\right) \leq \frac{1}{2}.$$

Hence, the median satisfies:

$$\boxed{m \leq \lambda + \frac{1}{3}}.$$

Chernoff bounds provide a method to control tail probabilities using exponential moments. They are especially useful in deriving bounds such as those for the median of the Poisson distribution.

Numerical Approximation Methods for the Median of a Negative Binomial Distribution

To find the exact median numerically, we avoid solving the intractable regularized incomplete beta function algebraically. Because the Negative Binomial distribution is discrete, the most efficient and standard programmatic approach is a **Mean-Anchored Cumulative Search**.

This numerical method uses the theoretical mean as a starting heuristic and applies the probability mass function (PMF) recurrence relation to find the smallest integer m where the Cumulative Distribution Function (CDF) crosses 0.5.

1. Mean-Anchored Cumulative Search

This algorithm minimizes computational complexity by avoiding large factorial calculations and starting the search very close to the true median.

Step 1: Calculate the initial guess
Compute the theoretical mean μ and round it to the nearest integer to use as our starting index k:

$$k = \left\lfloor \frac{r(1-p)}{p} \right\rceil$$

Step 2: Compute the initial PMF and CDF
Calculate the exact PMF, $P(X = k)$, and the CDF, $F(k)$, for this starting guess:

$$P(X = k) = \binom{k+r-1}{k} p^r (1-p)^k$$

$$F(k) = \sum_{j=0}^{k} P(X = j)$$

Step 3: Iterative Search via PMF Recurrence
To dynamically update the probability without recalculating the full summation or factorials, use the negative binomial recurrence relation.

- **Case A (Moving Forward):** If $F(k) < 0.5$, the median is larger than k. Iterate j upwards ($j = k, k+1, k+2, \dots$) and update the PMF and CDF at each step:

$$P(X = j+1) = P(X = j) \cdot \frac{j+r}{j+1}(1-p)$$

$$F(j+1) = F(j) + P(X = j+1)$$

Termination: Stop at the first integer m where $F(m) \geq 0.5$.

- **Case B (Moving Backward):** If $F(k) \geq 0.5$, k might be the median, or the true median might be smaller. Iterate j downwards ($j = k, k-1, k-2, \dots$) and reverse the calculation:

$$F(j-1) = F(j) - P(X = j)$$

$$P(X = j-1) = P(X = j) \cdot \frac{j}{(j+r-1)(1-p)}$$

  Termination: Stop at the exact step where $F(j-1) < 0.5$. The median m is the current j.

2. Bisection Method (Continuous Relaxation)

If a continuous root-finding method is required (for instance, to find a fractional median value before rounding it to a discrete integer), you can relax the integer constraint m to a continuous variable $x \in \mathbb{R}^+$.

We want to find the root x^* for the function $g(x)$:

$$g(x) = I_p(r, x+1) - 0.5 = 0$$

Step 1: Establish Brackets
Choose lower and upper bounds $[a, b]$ such that $g(a) < 0$ and $g(b) > 0$.

- Lower bound: $a = 0$
- Upper bound: $b = 2\left(\frac{r(1-p)}{p}\right)$ (Double the mean is a reliably safe upper bracket)

Step 2: Bisection Iteration
Repeat the following sequence until the interval width $(b - a)$ is less than your designated error tolerance ϵ (e.g., $\epsilon = 10^{-5}$):

1. Calculate the midpoint: $c = \frac{a+b}{2}$
2. Evaluate the function at the midpoint: $g(c) = I_p(r, c+1) - 0.5$
3. If $g(c) > 0$, the root lies in the lower half. Update the upper bound: $b = c$.
4. If $g(c) < 0$, the root lies in the upper half. Update the lower bound: $a = c$.

Step 3: Finalize the Discrete Median
Once the continuous root $x^* \approx c$ is identified, you map it back to the discrete Negative Binomial distribution. Because the discrete median requires the *smallest* integer that satisfies $F(m) \geq 0.5$, take the ceiling of the continuous root:

$$m = \lceil x^* \rceil$$