

March 24

Consider a discrete uniform RV 'X' with support

$$\{1, 2, 3\} = \Omega$$

i.e.,

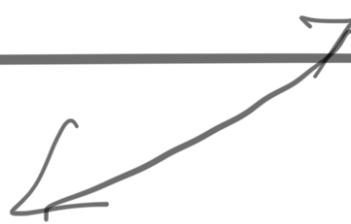
$$P(X=x) = \begin{cases} \frac{1}{3} & \text{if } x \in \Omega \\ 0 & \text{otherwise} \end{cases}$$

Consider a sample of size 2 from this pop.

$\{x_1, x_2\}$	Sample mean \bar{x}
$\{1, 1\}$	1
$\{1, 2\}$	1.5
$\{2, 1\}$	1.5
$\{2, 2\}$	2

$\{1, 3\}$	1
$\{2, 1\}$	1.5
$\{2, 2\}$	2
$\{2, 3\}$	2.5
$\{3, 1\}$	2
$\{3, 2\}$	2.5
$\{3, 3\}$	3

$$\text{Mean}(\bar{X}) = 2$$

$$E(X) = 2$$


Let X_1, \dots, X_n are

IID observations from
a distribution with mean
 μ and variance σ^2 .

We want to estimate

$$\hat{\mu}_1 = \frac{X_1 + \dots + X_n}{n} = \bar{X}$$
$$\hat{\mu}_2 = X_1, \text{ first obs.}$$

$$E(\hat{\mu}_1) = E(\bar{X}) = \mu$$

$$E(\hat{\mu}_2) = E(X_1) = \mu$$

So, $\hat{\mu}_1$ & $\hat{\mu}_2$ both
are unbiased for μ .

or unbi...

$$\begin{aligned} \text{Now } \text{Var}(\hat{\mu}_1) &= \text{Var}(\bar{X}) \\ &= \frac{\sigma^2}{n} \rightarrow 0 \end{aligned}$$

Chebyshev's ineq. \Rightarrow

$\hat{\mu}_1 \xrightarrow{p} \mu$ is consistent for μ .

But $\text{Var}(\hat{\mu}_2) = \text{Var}(X_1) = \sigma^2$

$$\mathbb{P}(|\hat{\mu}_2 - \mu| \leq \epsilon)$$

$$= \mathbb{P}(|X_1 - \mu| \leq \epsilon)$$

$$= \mathbb{P}(\mu - \epsilon \leq X_1 \leq \mu + \epsilon)$$

$$= \mathbb{P}(X_1 \in \underline{[\mu - \varepsilon, \mu + \varepsilon]})$$

$> \delta$ for some ε
even if $n \rightarrow \infty$

Let the distribution is
Bernoulli (μ); $\sigma^2 = \mu(1-\mu)$
 $\mu \in (0, 1) \rightarrow$ parameter space

$$\mathbb{P}(\underline{[\mu - \varepsilon \leq X_1 \leq \mu + \varepsilon]})$$

take $\varepsilon = \frac{\mu}{k}$, $k \in \mathbb{N}$
such that $\underline{\mu + \varepsilon} < 1$

$$= \mathbb{P}(X_1 = 0)$$

$$= 1 - \mu \longrightarrow \text{as } n \rightarrow \infty$$

$\Rightarrow \hat{\mu}_2 = x_1$, is not
a consistent estimator of μ .
But $\hat{\mu}_2$ is unbiased from μ .

Consistent but biased

In the same set up

$$\hat{\mu}_3 = \bar{X} + \frac{1}{n}$$

we know

$$\bar{X} \longrightarrow \mu$$

$$\left(\frac{1}{n} \right)$$

$\hat{\mu}_3 \xrightarrow{p} 0$ $\xrightarrow{p} \begin{pmatrix} \mu \\ 0 \end{pmatrix}$
 joint convergence + CMT

$$\bar{X} + \frac{1}{n} \xrightarrow{p} \mu + 0 = \mu$$

$$\Rightarrow \hat{\mu}_3 \xrightarrow{p} \mu$$

$$\text{Bias } (\hat{\mu}_3) = E(\hat{\mu}_3) - \mu$$

$$= E\left(\bar{X} + \frac{1}{n}\right) - \mu$$

$$= E(\bar{X}) + \frac{1}{n} - \mu$$

$$= \mu + \frac{1}{n} - \mu$$

$$= \frac{1}{n} \neq 0$$

 $\Rightarrow \hat{\mu}_3$ is not unbiased
for μ .

Consistency: An estimator

$\hat{\mu}$ for μ ;

(i) $\hat{\mu}$ is consistent for μ

if $\hat{\mu} \xrightarrow{p} \mu$
as $n \rightarrow \infty$
sample size

(ii) $\hat{\mu}$ is unbiased ^{for μ} if

$$E(\hat{\mu}) = \mu.$$

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