March 24

Consider a discrete uniform RV 'X' with support

$\{1, 2, 3\} = \Omega$

i.e,

$$\mathbb{P}(X = x) = \begin{cases} \frac{1}{3} & \text{if } x \in \Omega \\ 0 & \text{otherwise} \end{cases}$$

Consider a sample of size 2 from this pop.

$\{X_1, X_2\}$	sample mean $\bar{X}$
$\{1, 1\}$	1
$\{1, 2\}$	1.5
	2

$\{1,3\}$	2
$\{2,1\}$	1.5
$\{2,2\}$	2
$\{2,3\}$	2.5
$\{3,1\}$	2
$\{3,2\}$	2.5
$\{3,3\}$	3

$\text{Mean}(\bar{X}) = 2$

$\mathbb{E}(X) = 2$

Let $X_1, \ldots, X_n$ are

IID observations from a distribution with mean μ and variance σ^2.

We want to estimate μ.

$$\hat{\mu}_1 = \frac{X_1 + \dots + X_n}{n} = \bar{X}$$

$$\hat{\mu}_2 = X_1 \text{, first obs.}$$

$$\mathbb{E}(\hat{\mu}_1) = \mathbb{E}(\bar{X}) = \mu$$

$$\mathbb{E}(\hat{\mu}_2) = \mathbb{E}(X_1) = \mu$$

So, $\hat{\mu}_1$ & $\hat{\mu}_2$ both are unbiased for μ.

are unbiased ...

Now $\text{var}(\hat{\mu}_1) = \text{Var}(\bar{X})$

$$= \frac{\sigma^2}{n} \longrightarrow 0$$

Chebychev's ineq. $\Rightarrow$

$$\hat{\mu}_1 \xrightarrow{p} \mu.$$

$\Rightarrow \hat{\mu}_1$ is consistent for μ.

But $\text{Var}(\hat{\mu}_2) = \text{Var}(X_1) = \sigma^2$

$$\mathbb{P}(|\hat{\mu}_2 - \mu| \le \varepsilon)$$

$$= \mathbb{P}(|X_1 - \mu| \le \varepsilon)$$

$$= \mathbb{P}(\mu - \varepsilon \le X_1 \le \mu + \varepsilon)$$

$$= \mathbb{P}(X_1 \in [\mu-\varepsilon, \mu+\varepsilon])$$

$> \delta$ for some ε

even if $n \to \infty$

let the distribution is

Bernoulli (μ) ; $\sigma^2 = \mu(1-\mu)$

$\mu \in (0,1) \to$ parameter space

$$\mathbb{P}(\mu-\varepsilon \leq X_1 \leq \mu+\varepsilon)$$

take $\varepsilon = \mu/K$, $K \in \mathbb{N}$

such that $\mu+\varepsilon < 1$

$$= \mathbb{P}(X_1 = 0)$$

$= 1 - \mu \not\longrightarrow 1$ as $n \to \infty$

$\Rightarrow \hat{\mu}_2 = X_1$, is not a consistent estimator of μ.

But $\hat{\mu}_2$ is unbiased for μ.

Consistent but biased

In the same set up

$$\hat{\mu}_3 = \underbrace{\bar{X} + \frac{1}{n}}$$

we know

$\bar{X} \xrightarrow{P} \mu$

$$\frac{1}{n} \xrightarrow{p} 0 \quad \xrightarrow{p} \begin{pmatrix} \mu \\ 0 \end{pmatrix}$$

joint convergence + CMT

$$\bar{X} + \frac{1}{n} \xrightarrow{p} \mu + 0 = \mu$$

$$\Rightarrow \boxed{\hat{\mu}_3 \xrightarrow{p} \mu}$$

$$\text{Bias}(\hat{\mu}_3) = E(\hat{\mu}_3) - \mu$$

$$= E\left(\bar{X} + \frac{1}{n}\right) - \mu$$

$$= E(\bar{X}) + \frac{1}{n} - \mu$$

$$= \mu + \frac{1}{n} - \mu$$

$= \frac{1}{n} \neq 0$

$\Rightarrow \hat{\mu}_3$ is not unbiased for μ.

<u>Consistency</u>: An estimator $\hat{\mu}$ for μ;

(i) $\hat{\mu}$ is consistent for μ if $\hat{\mu} \xrightarrow{p} \mu$ as $n \to \infty$

n — sample size

(ii) $\hat{\mu}$ is unbiased for μ if

$E(\hat{\mu}) = \mu$.