

**Lecture notes  
Calculus**

**Chapter-1**

**Infinite Sequences and Series**

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## 1 Real number system

We are familiar with natural numbers and to some extent the rational numbers. While finding roots of algebraic equations we see that rational numbers are not enough to represent roots which are not rational numbers. For example draw the graph of  $y = x^2 - 2$ . We see that it cross the  $x$ -axis twice. The roots are such that their square is 2, but they cannot be rational numbers according to the following theorem.

**Theorem 1.0.1.** *Suppose that  $a_0, a_1, \dots, a_n (n \geq 1)$  are integers such that  $a_0 \neq 0, a_n \neq 0$  and that  $r$  satisfies the equation*

$$a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 = 0.$$

*If  $r = \frac{p}{q}$  where  $p, q$  are integers with no common factors and  $q \neq 0$ . Then  $q$  divides  $a_n$  and  $p$  divides  $a_0$ .*

*Proof.* Since  $\frac{p}{q}$  satisfies the equation, we have

$$a_n p^n + a_{n-1} p^{n-1} q + \dots + a_0 q^n = 0$$

i.e.,  $a_n p^n = -q(a_{n-1} p^{n-1} + \dots + a_0 q^{n-1})$ . This means  $q$  divides  $a_n$  as  $p, q$  have no common factors. On the other hand we can also write

$$a_0 q^n = -p(a_n p^{n-1} + a_{n-1} p^{n-2} q + \dots + a_1 q^{n-1}).$$

Thus  $p$  divides  $a_0$ . (Here we used the following:  $p, q$  can be expressed as  $p = p_1 \dots p_j$  and  $q = q_1 \dots q_k$  where each  $p_i, q_i$  are prime numbers. Since  $p$  divides  $a_0 q^n$ , the quantity

$$\frac{a_0 q^n}{p} = a_0 \frac{q_1^n \dots q_k^n}{p_1 \dots p_j}$$

must be an integer. Since no  $p_i$  is equal to  $q_j$ , the unique prime factorization of  $a_0$  must include the product  $p_1 \dots p_j$ .)

As a consequence, we see that the possible rational roots of  $x^2 - 2 = 0$  are  $\pm 1, \pm 2$ . But it is easy to check that  $\pm 1, \pm 2$  does not satisfy  $x^2 - 2 = 0$ . So the roots of  $x^2 - 2 = 0$  are not rational numbers. Another example,

**Problem 1.** *Show that  $a = (2 + \sqrt{2})^{1/2}$  is not a rational number.*

Easy to see that  $a$  satisfies the polynomial

$$a^4 - 4a^2 + 2 = 0$$

If a rational number  $r = \frac{p}{q}$  is a root then

$$p \in \{\pm 1, \pm 2\} \text{ and } q \in \{\pm 1\}.$$

Such a number cannot be root of the above polynomial.

This means the set of rational numbers has "gaps". So the natural question to ask is: Can we have a number system without these gaps? The answer is yes and the "complete number system" without these gaps is the real line  $\mathbb{R}$ . We will not look into the development of  $\mathbb{R}$  as it is not easy to define the real numbers. We assume that there is a set  $\mathbb{R}$ , whose elements are called real numbers and  $\mathbb{R}$  is closed with respect to addition and multiplication. That is, given any  $a, b \in \mathbb{R}$ , the sum  $a + b$  and product  $ab$  also represent real numbers. Moreover,  $\mathbb{R}$  has an order structure  $\leq$  and has no "gaps" in the sense that it satisfies the Completeness Axiom (see below).

An application of the rational number theorem is

**Problem 2.** Find the roots of  $x^3 - 2x^2 - 2x + 4 = 0$ .

By rational number theorem the only rationals that are possibly roots are  $\pm 2, \pm 4$ . Easy to check that  $x = 2$  satisfies the equation. So by division algorithm, we get

$$x^3 - 2x^2 - 2x + 4 = (x - 2)(x^2 - 2).$$

Other two roots are  $\pm\sqrt{2}$ .

Let  $S$  be a non-empty subset of  $\mathbb{R}$ . If  $S$  contains a largest element  $s^0$ , then we call  $s^0$  the maximum of  $S$ . If  $S$  contains a smallest element  $s_0$ , then we call  $s_0$  the minimum of  $S$ . If  $S$  is bounded above and  $S$  has least upper bound  $\bar{s}$ , then we call this  $\bar{s}$ , the supremum of  $S$ . If  $S$  is bounded below and  $S$  has greatest lower bound  $\underline{s}$ , then we call  $\underline{s}$ , the infimum of  $S$ .

**Problem 3.** Find the supremum of the set  $A = \{x \in \mathbb{Q} : 0 < (\sqrt{5} - 2)x < \sqrt{5} + 2\}$ .

If  $x \in A$  then

$$x < \frac{\sqrt{5} + 2}{\sqrt{5} - 2} = 9 + 4\sqrt{5}$$

Here we notice that  $9 + 4\sqrt{5}$  is not a rational number (why?). Hence the supremum is not achieved in  $A$ .

Unlike maximum and minimum,  $\sup S$  and  $\inf S$  need not belong to the set  $S$ . An important observation is if  $\alpha = \sup S$  is finite, then for every  $\epsilon > 0$ , there exists an element  $s \in S$  such that  $s \geq \alpha - \epsilon$ .

Note that any bounded subset of Natural numbers has maximum and minimum.

**Completeness Axiom:** Every nonempty subset  $S$  of  $\mathbb{R}$  that is bounded above has a least upper bound. In other words,  $\sup S$  exists and is a real number.

The completeness axiom does not hold for  $\mathbb{Q}$ . That is, every non-empty subset of  $\mathbb{Q}$  that is bounded above by a rational number need not have rational least upper bound. For example

**Example 1.0.2.** The maximum of the set  $S = \{r \in \mathbb{Q} : r^2 \leq 2\}$  is not a rational number.

Suppose  $p \in S$ , then define

$$q = p + \frac{p^2 - 2}{p + 2}$$

then  $q > p$  and it is easy to verify

$$q^2 - 2 = \frac{2(p^2 - 2)}{(p + 2)^2}$$

Hence  $q \in S$ . So for every  $p \in S$ , there exists  $q \in S$  and  $q > p$ . Therefore  $S$  does not have a maximum.

**Archimedean property:**

**Theorem 1.0.3.** *For each  $x \in \mathbb{R}$ , there exists a natural number  $N = N(x)$  such that  $x < N$ . That is, the set of natural numbers is unbounded (from above) in  $\mathbb{R}$ .*

*Proof.* Assume by contradiction that this is not true. Then there is no  $N \in \mathbb{N}$  such that  $x < N$ . i.e.,  $x$  is an upper bound for  $\mathbb{N}$ . Then, by completeness axiom, let  $u$  be the smallest such bound of  $\mathbb{N}$  in  $\mathbb{R}$ . That is  $u \in \mathbb{R}$  and so  $u - m$  for  $2 \leq m \in \mathbb{N}$  is not an upper bound for  $\mathbb{N}$ . Therefore, there exists  $k \in \mathbb{N}$  such that  $u - m < k$ , but then  $u < k + m$ , and  $k + m \in \mathbb{N}$ . a contradiction.

Now it is easy to see the following corollary

**Corollary 1.0.4.** *Let  $S = \{\frac{1}{n} : n \in \mathbb{N}\}$ . Then  $w = \inf S = 0$ .*

*Proof.* We note that  $S$  is bounded below. Let  $\epsilon > 0$  be an arbitrary positive real number. By above Archimedean property, there exists  $n \in \mathbb{N}$  such that  $n > \frac{1}{\epsilon}$ . Then we have,

$$0 \leq w \leq \frac{1}{n} < \epsilon.$$

Since  $\epsilon$  is arbitrary, we have  $w = 0$ . (why?)

**Corollary 1.0.5.** *If  $y > 0$  be a real number, then there exists  $n = n(y) \in \mathbb{N}$  such that  $n - 1 \leq y < n$ .*

**Problem 4.** *find the infimum and supremum of the set of real numbers  $A = \left\{ \frac{n}{n+1}, n \in \mathbb{N} \right\}$*

Easy to see that

$$n \leq m \iff \frac{n}{n+1} \leq \frac{m}{m+1}$$

Therefore set can be ordered in increasing order. Then it is easy to see that

$$\inf A = \min A = \frac{1}{2}.$$

It is again easy to see that  $\frac{n}{n+1} \leq 1$ . So 1 is an upper bound. We claim that 1 is the supremum. Suppose not. Then there exists  $s < 1$  such that  $\frac{n}{n+1} \leq s$  for all  $n \in \mathbb{N}$ . But this immediately

implies

$$n \leq \frac{s}{1-s}, \text{ for all } n \in \mathbb{N}.$$

On the other hand, by algebraic properties of real numbers  $\frac{s}{1-s}$  is a real number and by archimedean property there is at least one natural number  $N_0$  such that  $N_0 > \frac{s}{1-s}$ , a contradiction.

Finally, we have the following density theorem

**Theorem 1.0.6.** *Let  $x, y$  are real numbers such that  $x < y$ . Then there exists a rational number  $q$  such that  $x < q < y$ .*

**Proof:** Without loss of generality, assume that  $x > 0$ . Now let  $n \in \mathbb{N}$  be such that  $y - x > \frac{1}{n}$  (Archimedean property). Now consider the set

$$S = \{m \in \mathbb{N} : \frac{m}{n} > x\}.$$

Then  $S$  is non-empty (by Archimedean property). By well-ordering of  $\mathbb{N}$ ,  $S$  has minimal element say  $m_0$ . Then  $x < \frac{m_0}{n}$ . By the minimality of  $m_0$ , we see that  $\frac{m_0-1}{n} \leq x$ . Then,

$$\frac{m_0}{n} \leq x + \frac{1}{n} < x + (y - x) = y.$$

Therefore,

$$x < \frac{m_0}{n} < y.$$

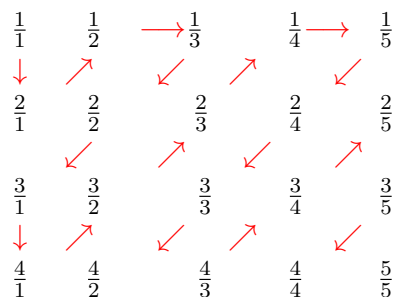
**Definition 1.0.7.** *We say a subset  $E$  of  $\mathbb{R}$  is countable if either  $E$  is finite or there is a bijection between  $\mathbb{N}$  and  $E$ .*

We can define a map from  $\mathbb{N}$  to  $\mathbb{Q}$  following some order. On the otherhand it can be shown that countable union of finite sets is also countable(easy to see this).

**Remark 1.1.**  $\mathbb{Q}$  is countable. Indeed, for each  $n \in \mathbb{N}$ , define the set

$$E_n = \left\{ 0 < r \in \mathbb{Q}, r = \frac{p}{q}, p + q = n \right\}$$

For example  $E_2 = \{\frac{1}{1}\}$ ,  $E_3 = \{\frac{2}{1}, \frac{1}{2}\}$ ,  $E_4 = \{\frac{3}{1}, \frac{2}{2}, \frac{1}{3}\}$ . Each  $E_n$  contains finitely many elements and  $\mathbb{Q}^+ = \cup_n E_n$ . Hence  $\mathbb{Q}$  is countable. Indeed, we have the 1-1, onto map:



## 2 Infinite Sequences

We are familiar with finite sequences and some infinite sequences like arithmetic and geometric sequences. Another important sequence of numbers is due to Acharya Pingala. While exploring the number of possibilities of various combinations of the laghu swar (single matra) and the guru swar (two matras), Acharya Pingala (3rd-2nd BCE) hit upon a sequence of numbers:  $\{0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \dots\}$ . These combinations were used in understanding and analyzing the rhythmic and structural elements within Sanskrit poetry. This sequence later became popular as Fibonacci sequence where each number is sum of previous two numbers. This appeared in the book Liber Abaci (1202 AD) by Fibonacci where the Indian number system is introduced in the west to replace Roman number system. This Fibonacci sequence has wide applications in the study of algorithms.

In the previous section we observed that there are real numbers which may not have exact expression like rational numbers. One way to find a nearby rational number is to use the rational number theorem. If  $x$  is a given real number that is not rational, then we take a rational  $r_1$  and we now know there is another rational that lies between  $r_1$  and  $x$ , say  $r_2$ . By proceeding this way we get an infinite collection of rationals  $r_n$  that are very near to  $x$  as  $n$  is taken large enough. So we are naturally interested in the study of sequences that approach a real number.

### Application: Analyzing the data

The stock price of a company or temperature at 12 noon on each day over the years is a data that forms a sequence. By analyzing this sequence one tries to derive certain conclusions, say the indication for global warming. Even average rain fall over 100 years etc.. Below is a graph that is generated with the closing stock price of Infosys over one year.



Figure 1: Infosys Stock price

## 2.1 Limit of sequence

**Definition 2.1.1.** A sequence of real numbers is a function from  $\mathbb{N}$  to  $\mathbb{R}$ . A sequence is denoted as  $\{a_n\}_{n=1}^{\infty}$ . That is,

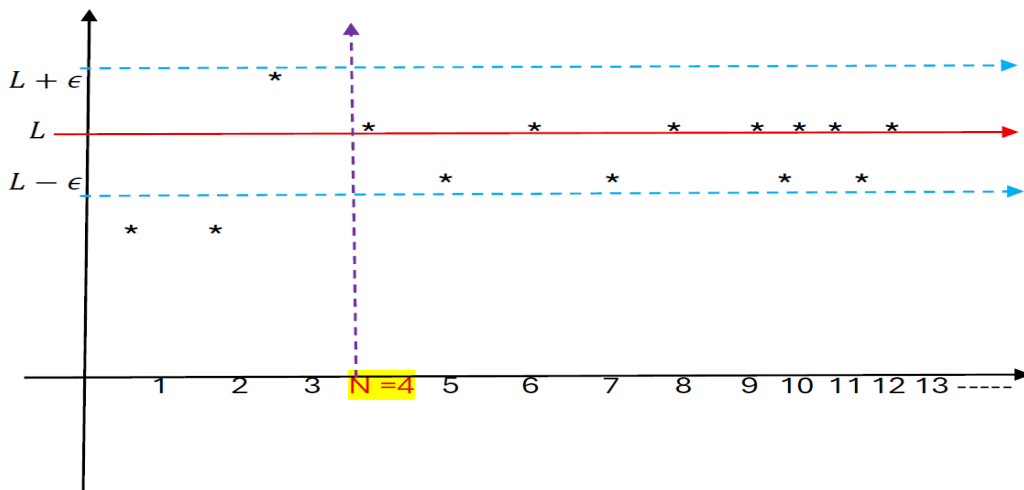
$$\{a_n\}_{n=1}^{\infty} = \{a_1, a_2, a_3, \dots\}$$

**Examples 2.1.2.** (i)  $\{c\}_{n=1}^{\infty}, c \in \mathbb{R}$ , (ii)  $\{\frac{(-1)^{n+1}}{n}\}_{n=1}^{\infty}$ , (iii)  $\{\frac{n-1}{n}\}_{n=1}^{\infty}$  and (iv)  $\{\sqrt{n}\}_{n=1}^{\infty}$ .

**Definition 2.1.3.** A sequence  $\{a_n\}_{n=1}^{\infty}$  converges to limit  $L$  if for every  $\epsilon > 0$  (given) there exists a positive integer  $N$  such that

$$n \geq N \implies |a_n - L| < \epsilon.$$

In this case, we write  $L = \lim_{n \rightarrow \infty} a_n$  or  $a_n \rightarrow L$  as  $n \rightarrow \infty$



**Examples 2.1.4.**

(i) It is clear that the constant sequence  $\{c\}_{n=1}^{\infty}, c \in \mathbb{R}$ , has  $c$  as its limit.

(ii) Show that  $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$ .

**Solution:** Let  $\epsilon > 0$  be given. In order to show that  $1/n$  approaches 0, we must show that there exists an integer  $N \in \mathbb{N}$  such that for all  $n \geq N$ ,

$$\left| \frac{1}{n} - 0 \right| = \frac{1}{n} < \epsilon.$$

But

$$\frac{1}{n} < \epsilon \iff n > \frac{1}{\epsilon}.$$

By Archimedean property, there exists  $N \in \mathbb{N}$  such that  $N > 1/\epsilon$ . Then for all  $n \geq N$ , we have  $\frac{1}{n} < \epsilon$ .

(iii) Consider the sequence  $\{(-1)^{n+1}\}_{n=1}^{\infty}$ . When we plot the sequence, we see that all the even indexed elements lie on the line  $L = -1$  and odd indexed elements lie on the line  $L = 1$ . So we if take any line  $L$  and  $\epsilon = 1/2$ , All the elements cannot lie in the strip of width 1. So it is intuitively clear that this sequence does not have a limit. We now prove this by definition. Assume to the contrary, that there exists an  $L \in \mathbb{R}$  such that the sequence  $\{(-1)^{n+1}\}_{n=1}^{\infty}$  converges to  $L$ . Then for  $\epsilon = \frac{1}{2}$ , there exists an  $N \in \mathbb{N}$  such that

$$|(-1)^{n+1} - L| < \frac{1}{2}, \forall n \geq N. \quad (2.1)$$

For  $n$  even, (2.1) says

$$|-1 - L| < \frac{1}{2}, \forall n \geq N. \quad (2.2)$$

while for  $n$  odd, (2.1) says

$$|1 - L| < \frac{1}{2}, \forall n \geq N. \quad (2.3)$$

which is a contradiction as  $2 = |1 + 1| \leq |1 - L| + |1 + L| < 1$ .

As a first result we have the following uniqueness theorem:

**Theorem 2.1.5.** *If  $\{a_n\}_1^{\infty}$  is a sequence and if both  $\lim_{n \rightarrow \infty} a_n = L$  and  $\lim_{n \rightarrow \infty} a_n = M$  holds, then  $L = M$ .*

*Proof.* Suppose that  $L \neq M$ . Then  $|L - M| > 0$ . Let  $\epsilon = \frac{|L-M|}{4}$ . As  $\lim_{n \rightarrow \infty} a_n = L$ , there exists  $N_1 \in \mathbb{N}$  such that

$$|a_n - L| < \epsilon \text{ for all } n \geq N_1.$$

Also as  $\lim_{n \rightarrow \infty} a_n = M$ , there exists  $N_2 \in \mathbb{N}$  such that

$$|a_n - M| < \epsilon \text{ for all } n \geq N_2.$$

Let  $N = \max\{N_1, N_2\}$ . Then for all  $n \geq N$ ,  $|a_n - L| < \epsilon$  and  $|a_n - M| < \epsilon$ . Thus

$$|L - M| \leq |a_n - L| + |a_n - M| < 2\epsilon = \frac{1}{2}|L - M|,$$

which is a contradiction.

**Definition 2.1.6. Bounded sequence:** A sequence  $\{a_n\}$  is said to be bounded above, if there exists  $M \in \mathbb{R}$  such that  $a_n \leq M$  for all  $n \in \mathbb{N}$ . Similarly, we say that a sequence  $\{a_n\}$  is bounded below, if there exists  $N \in \mathbb{R}$  such that  $a_n \geq N$  for all  $n \in \mathbb{N}$ . Thus a sequence  $\{a_n\}$  is said to be bounded if it is both bounded above and below.

**Examples 2.1.7.** 1.  $\{\frac{n+1}{n}\}$  is bounded below by 0 and  $\frac{n+1}{n} \leq \frac{2n}{n} = 2$  implies the sequence is bounded above by 2

2.  $\{\frac{(-1)^{n+1}}{n}\}$  is bounded above by 1 and bounded below by  $-\frac{1}{2}$ .

**Theorem 2.1.8.** Every convergent sequence is bounded.

*Proof.* Let  $\{a_n\}$  be a convergent sequence and  $L = \lim_{n \rightarrow \infty} a_n$ . Let  $\epsilon = 1$ . Then there exists  $N \in \mathbb{N}$  such that  $|a_n - L| < 1$  for all  $n \geq N$ . Further,

$$|a_n| = |a_n - L + L| \leq |a_n - L| + |L| < 1 + |L|, \forall n \geq N.$$

Let  $M = \max\{|a_1|, |a_2|, \dots, |a_{N-1}|, 1 + |L|\}$ . Then  $|a_n| \leq M$  for all  $n \in \mathbb{N}$ .

## 2.2 Algebra of convergent sequences

**Theorem 2.2.1.** Let  $\{a_n\}_1^\infty$  and  $\{b_n\}_1^\infty$  be two sequences such that  $\lim_{n \rightarrow \infty} a_n = L$  and  $\lim_{n \rightarrow \infty} b_n = M$ . Then

$$(i) \quad \lim_{n \rightarrow \infty} (a_n + b_n) = L + M.$$

$$(ii) \quad \lim_{n \rightarrow \infty} (ca_n) = cL, \quad c \in \mathbb{R}.$$

$$(iii) \quad \lim_{n \rightarrow \infty} (a_n b_n) = LM.$$

$$(iv) \quad \lim_{n \rightarrow \infty} \left( \frac{1}{a_n} \right) = \frac{1}{L} \text{ if } L \neq 0.$$

*Proof.*

(i) Let  $\epsilon > 0$ . Since  $a_n$  converges to  $L$  there exists  $N_1 \in \mathbb{N}$  such that

$$|a_n - L| < \epsilon/2 \quad \forall n \geq N_1.$$

Also, as  $b_n$  converges to  $M$  there exists  $N_2 \in \mathbb{N}$  such that

$$|b_n - M| < \epsilon/2 \quad \forall n \geq N_2.$$

Thus for all  $n \geq N = \max\{N_1, N_2\}$

$$|(a_n + b_n) - (L + M)| \leq |a_n - L| + |b_n - M| < \epsilon.$$

(ii) Easy to prove. Hence left as an exercise.

(iii) Let  $\epsilon > 0$ . Since  $a_n$  is a convergent sequence, it is bounded by  $M_1$  (say). Also as  $a_n$  converges to  $L$  there exists  $N_1 \in \mathbb{N}$  such that

$$|a_n - L| < \epsilon/2M_1 \quad \forall n \geq N_1.$$

Similarly as  $b_n$  converges to  $M$  there exists  $N_2 \in \mathbb{N}$  such that

$$|b_n - M| < \epsilon/2M_1 \quad \forall n \geq N_2.$$

Let  $N = \max\{N_1, N_2\}$ . Then

$$\begin{aligned} |a_n b_n - LM| &= |a_n b_n - a_n M + a_n M - LM| \leq |a_n(b_n - M)| + |M(a_n - L)| \\ &= |a_n||b_n - M| + M|a_n - L| < \epsilon/2 + \epsilon/2 = \epsilon \end{aligned}$$

(iv) Without loss of generality, let us assume that  $L > 0$ . Let  $\epsilon > 0$  be given. As  $\{a_n\}$  forms a convergent sequence, it is bounded. Choose  $N_1 \in \mathbb{N}$  such that  $a_n > L/2$  for all  $n \geq N_1$ . Also, as  $a_n$  converges to  $L$ , there exists  $N_2 \in \mathbb{N}$  such that

$$|a_n - L| < \frac{L^2 \epsilon}{2} \text{ for all } n \geq N_2.$$

Let  $N = \max\{N_1, N_2\}$ . Then

$$n \geq N \implies \left| \frac{1}{a_n} - \frac{1}{L} \right| = \frac{|a_n - L|}{|a_n L|} < \frac{2}{L^2} \frac{L^2 \epsilon}{2} = \epsilon.$$

### Examples 2.2.2.

(i) Consider the sequence  $\left\{ \frac{5}{n^2} \right\}_1^\infty$ . By (ii)&(iii) of theorem 2.2.1, we get

$$\lim_{n \rightarrow \infty} \frac{5}{n^2} = \lim_{n \rightarrow \infty} 5 \cdot \frac{1}{n} \cdot \frac{1}{n} = 5 \cdot \lim_{n \rightarrow \infty} \frac{1}{n} \cdot \lim_{n \rightarrow \infty} \frac{1}{n} = 5 \cdot 0 \cdot 0 = 0.$$

(ii) Consider the sequence  $\left\{ \frac{3n^2 - 6n}{5n^2 + 4} \right\}_1^\infty$ .

Notice that  $\frac{3n^2 - 6n}{5n^2 + 4} = \frac{3 - \frac{6}{n}}{5 + \frac{4}{n}}$ . Since  $5 + \frac{4}{n} \rightarrow 5 \neq 0$  by (i) – (iv) of theorem 2.2.1, we get

$$\lim_{n \rightarrow \infty} \frac{3n^2 - 6n}{5n^2 + 4} = \lim_{n \rightarrow \infty} \frac{3 - \frac{6}{n}}{5 + \frac{4}{n}} = \frac{3 - \lim_{n \rightarrow \infty} \frac{6}{n}}{5 + \lim_{n \rightarrow \infty} \frac{4}{n}} = \frac{3}{5}.$$

### Theorem 2.2.3. Sandwich theorem for sequences:

Let  $\{a_n\}$ ,  $\{b_n\}$  and  $\{c_n\}$  be three sequences such that

1.  $a_n \leq b_n \leq c_n$  for all  $n \in \mathbb{N}$

2.  $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} c_n = L$ .

Then  $\lim_{n \rightarrow \infty} b_n = L$ .

*Proof.* Let  $\epsilon > 0$  be given. As  $\lim_{n \rightarrow \infty} a_n = L$ , there exists  $N_1 \in \mathbb{N}$  such that

$$n \geq N_1 \implies |a_n - L| < \epsilon. \quad (2.4)$$

Similarly as  $\lim_{n \rightarrow \infty} c_n = L$ , there exists  $N_2 \in \mathbb{N}$

$$n \geq N_2 \implies |c_n - L| < \epsilon. \quad (2.5)$$

Let  $N = \max\{N_1, N_2\}$ . Then,  $L - \epsilon < a_n$  (from (2.4)) and  $c_n \leq L + \epsilon$  (from (2.5)). Thus

$$L - \epsilon < a_n \leq b_n \leq c_n \leq L + \epsilon.$$

Thus  $|b_n - L| < \epsilon$  for all  $n \geq N$ . Hence the proof.

#### Examples 2.2.4.

(i) Consider the sequence  $\left\{\frac{\cos n}{n}\right\}_{n=1}^{\infty}$ . Then  $\frac{-1}{n} \leq \frac{\cos n}{n} \leq \frac{1}{n}$ . Since the sequences  $\{\frac{1}{n}\}$  and  $\{\frac{-1}{n}\}$  converge to 0, by Sandwich theorem 2.2.3,  $\lim_{n \rightarrow \infty} \frac{\cos n}{n} = 0$ .

(ii) As  $0 \leq \frac{1}{2^n} \leq \frac{1}{n}$  and  $\frac{1}{n} \rightarrow 0$  as  $n \rightarrow \infty$ , implies the sequence  $\{\frac{1}{2^n}\}$  also converges to 0 by Sandwich theorem 2.2.3.

**Problem 5.** For any  $b > 0$ , the sequence  $a_n := b^{\frac{1}{n}} \rightarrow 1$  as  $n \rightarrow \infty$ .

**Solution:** First assume that  $b > 1$ . Let  $a_n = b^{\frac{1}{n}} - 1$ . As  $b > 1$ ,  $a_n > 0$  for all  $n \in \mathbb{N}$ . Further,

$$b = (1 + a_n)^n \geq 1 + na_n.$$

Then  $0 \leq a_n \leq \frac{b-1}{n}$ . Thus  $a_n \rightarrow 0$ , i.e.,  $b^{\frac{1}{n}} \rightarrow 1$  as  $n \rightarrow \infty$ .

Now if  $b < 1$ , then take  $c = \frac{1}{b}$  and it is easy to show the result.

**Problem 6.** Prove the following:

(i)  $\lim_{n \rightarrow \infty} \sqrt[n]{n} = 1$ .

(ii) If  $p > 0$  then  $\lim_{n \rightarrow \infty} \frac{n^p}{(1+p)^n} = 0$ .

(iii) If  $p > 0$ , then  $\lim_{n \rightarrow \infty} \frac{\log(n)}{n^p} = 0$ .

#### Solution

(i) Let  $a_n = n^{\frac{1}{n}} - 1$ . Then

$0 \leq a_n \leq 1$  for all  $n \in \mathbb{N}$ . Further,

$$n = (1 + a_n)^n \geq \frac{n(n-1)}{2} a_n^2.$$

Thus  $0 \leq a_n \leq \sqrt{\frac{2}{(n-1)}}$  ( $n \geq 2$ ). As  $\sqrt{\frac{2}{(n-1)}} \rightarrow 0$  as  $n \rightarrow \infty$ , by Sandwich theorem,  $a_n \rightarrow 0$ , i.e.,  $n^{\frac{1}{n}} \rightarrow 1$  as  $n \rightarrow \infty$ .

(ii) Let  $k$  be an integer such that  $k > p$ ,  $k > 0$ . Then for  $n > 2k$ ,

$$(1+p)^n > {}^nC_k p^k = \frac{n!}{k!(n-k)!} p^k = \frac{p^k}{k!} \prod_{i=1}^k [n-i+1] > \frac{n^k}{2^k} \frac{p^k}{k!}.$$

Hence,

$$0 < \frac{n^p}{(1+p)^n} < \frac{2^k k!}{p^k} n^{p-k}.$$

As  $p - k < 0$ ,  $n^{p-k} \rightarrow 0$ . Thus  $\frac{n^p}{(1+p)^n} \rightarrow 0$  as  $n \rightarrow \infty$ .

- (iii) First we take the case  $p = 1$ . Let  $\epsilon > 0$ . From the previous problem we have  $n^{\frac{1}{n}} \rightarrow 1$  as  $n \rightarrow \infty$ . From the definition of logarithm

$$n^{\frac{1}{n}} = t \iff \frac{\log n}{n} = \log t$$

So  $|n^{\frac{1}{n}} - 1| = |t - 1|$  is small  $\iff |\frac{\log n}{n}| = |\log t - 1|$  is small.

The case when  $p > 1$ , we notice that

$$0 \leq \frac{\log n}{n^p} < \frac{\log n}{n} \rightarrow 0$$

For the case  $0 < p < 1$ , by the Archimedian property, for any  $n \in \mathbb{N}$  there exists  $m \in \mathbb{N}$  such that

$$m \leq n^p < (m+1)$$

This implies

$$\log n < \frac{1}{p} \log(m+1)$$

Therefore,

$$0 \leq \frac{\log n}{n^p} < \frac{1}{p} \frac{\log(m+1)}{m} = \frac{1}{p} \frac{\log(m+1)}{m+1} \left( \frac{m+1}{m} \right) \rightarrow \frac{1}{p} \cdot 0 \cdot 1.$$

The conclusion follows from case  $p = 1$  and Sandwich theorem.

**Definition 2.2.5. Subsequence:** Let  $\{a_n\}$  be a sequence and  $\{n_1, n_2, \dots\}$  be a sequence of positive integers such that  $i > j$  implies  $n_i > n_j$ . Then the sequence  $\{a_{n_i}\}_{i=1}^{\infty}$  is called a subsequence of  $\{a_n\}$ .

For example, if the depth of a river at every 10 meters form a sequence then the depth at every 100 meters forms its sub-sequence.

**Example 2.2.6.** Consider the sequence  $\{\cos n\pi\}$ . If we take  $n_k = 2k$  we get the subsequence  $\{\cos 2k\pi\} = \{1, 1, 1, \dots\}$ . If we take  $n_k = 2k+1$  then we get the subsequence  $\{\cos(2k+1)\pi\} = \{-1, -1, -1, \dots\}$ .

**Theorem 2.2.7.** If the sequence of real numbers  $\{a_n\}_1^{\infty}$ , is convergent to  $L$ , then any subsequence of  $\{a_n\}$  is also convergent to  $L$ .

*Proof.* Let  $\{n_i\}_{i=1}^{\infty}$  be a sequence of positive integers such that  $\{a_{n_i}\}_{i=1}^{\infty}$  forms a subsequence of  $\{a_n\}$ . Let  $\epsilon > 0$  be given. As  $\{a_n\}$  converges to  $L$ , there exists  $N \in \mathbb{N}$  such that

$$|a_n - L| < \epsilon, \quad \forall n \geq N.$$

Choose  $M \in \mathbb{N}$  such that  $n_i \geq N$  for  $i \geq M$ . Then

$$|a_{n_i} - L| < \epsilon, \quad \forall i \geq M.$$

Hence the proof.

**Problem 7.** *Prove the following*

1. *If a sequence  $\{a_n\}$  converges to  $a$ . Then  $\{|a_n|\}$  converges to  $|a|$ .*
2. *If  $a_n \geq 0$  and  $a_n \rightarrow a$ , then  $\{\sqrt{a_n}\}$  converges to  $\sqrt{a}$ .*

**Solution:** Proof of (1) follows from the inequality

$$||a| - |b|| \leq |a - b|, \quad \forall a, b \in \mathbb{R}.$$

(2) follows from the fact that if  $a \neq 0$ , then

$$|\sqrt{a_n} - \sqrt{a}| \leq \frac{|a_n - a|}{|\sqrt{a_n} + \sqrt{a}|}.$$

The case  $a = 0$  is easy and is left as an exercise.

**Problem 8.** *If a sequence  $\{a_n\}$  converges to  $a$ , then  $\left\{\frac{1}{n} \sum_{k=1}^n a_k\right\}$  also converges to  $a$ .*

Intuitively, the new sequence is the sequence of averages. We leave this as an exercise.

Next we study some fine properties of sequences that imply convergence/divergence.

**Theorem 2.2.8.** *For any sequence  $\{a_n\}$  with  $a_n > 0$*

$$\lim_{n \rightarrow \infty} a_n^{\frac{1}{n}} = \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n}$$

*provided the limit on the right side exists.*

*Proof.* Let  $\epsilon > 0$  be arbitrary. Suppose the second limit exists (say  $l$ ), then there exists  $N \in \mathbb{N}$  such that

$$l - \epsilon < \frac{a_{n+1}}{a_n} < l + \epsilon, \quad \forall n \geq N.$$

Taking  $n = N, N + 1, \dots, m - 1$  and multiplying we get

$$(l - \epsilon)^{m-N} < \frac{a_m}{a_N} < (l + \epsilon)^{m-N}, \quad \forall m \geq N + 1$$

equivalently,

$$(l - \epsilon)^{1 - \frac{N}{m}} a_N^{\frac{1}{m}} < (a_m)^{\frac{1}{m}} < (l + \epsilon)^{1 - \frac{N}{m}} a_N^{\frac{1}{m}}, \quad \forall m \geq N + 1.$$

Now since  $\lim_{m \rightarrow \infty} (l \pm \epsilon)^{1 - N/m} a_N^{1/m} = l \pm \epsilon$ , we get  $m_1, m_2$  such that

$$l - 2\epsilon < (l - \epsilon)^{1 - \frac{N}{m}} a_N^{\frac{1}{m}}, \quad \forall m \geq m_1$$

$$(l + \epsilon)^{1 - \frac{N}{m}} a_N^{\frac{1}{m}} < l + 2\epsilon, \quad \forall m \geq m_2$$

therefore for all  $m \geq \max\{m_1, m_2\}$

$$l - 2\epsilon < (a_m)^{\frac{1}{m}} < l + 2\epsilon$$

Hence the result. (If  $l = 0$ , then  $l - \epsilon$  is negative. In this case it is sufficient to take Right hand side of above inequalities)

**Corollary 2.2.9.** *Above theorem holds even if the limit on the Right hand side approaches infinity (defined below).*

**Corollary 2.2.10.** *Show the following:*

(i) If  $a_n > 0$  and  $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = l < 1$ , then  $\lim_{n \rightarrow \infty} a_n = 0$ .

(ii) If  $a_n > 0$  and  $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = l > 1$ , then  $a_n \rightarrow \infty$ .

*Proof.* If  $l < 1$ , then choose  $\epsilon_0$  such that  $l + \epsilon_0 < 1$ . Then for this  $\epsilon_0$  there exists  $N_0$  such that

$$n \geq N_0 \implies \frac{a_{n+1}}{a_n} < l + \epsilon_0$$

Therefore from the hypothesis of (i), there exists  $N_1$  such that

$$n \geq N_1 \implies a_n^{\frac{1}{n}} < l + \epsilon_0$$

Therefore  $a_n < (l + \epsilon_0)^n \rightarrow 0$  as  $n \rightarrow \infty$ . A similar argument will imply the (ii).

**Problem 9.** *Show the following:*

$$(1) \lim_{n \rightarrow \infty} a^{\frac{1}{n}} = 1, \text{ if } a > 0 \quad (ii) \lim_{n \rightarrow \infty} n^\alpha x^n = 0, \text{ if } |x| < 1 \text{ and } \alpha \in \mathbb{R}.$$

**Solution:**

(i) Take  $a_n = a$ , then  $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = 1$ .

(ii) If  $x \neq 0$ , take  $a_n = n^\alpha x^n$ , then  $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^\alpha |x| = |x|$ .

**Problem 10.** *Find the limit of the sequence  $\left\{ \frac{2^n + 3^n}{4^n + 5^n} \right\}_{n=1}^\infty$*

It is not difficult to see that

$$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \frac{3}{5} \lim_{n \rightarrow \infty} \frac{[(\frac{4}{5})^n + 1]}{[(\frac{4}{5})^{n+1} + 1]} \frac{[(\frac{2}{3})^{n+1} + 1]}{[(\frac{2}{3})^n + 1]} = \frac{3}{5} < 1.$$

Therefore,  $\lim_{n \rightarrow \infty} a_n = 0$ .

**Definition 2.2.11.** Let  $\{a_n\}$  be a sequence of real numbers. We say that  $a_n$  approaches infinity or diverges to infinity, if for any real number  $M > 0$ , there is a positive integer  $N$  such that

$$n \geq N \implies a_n \geq M.$$

(a) If  $a_n$  approaches infinity, then we write  $a_n \rightarrow \infty$  as  $n \rightarrow \infty$ .

(b) A similar definition is given for the sequences diverging to  $-\infty$ .

**Examples 2.2.12.**

(i) The sequence  $\{n^2\}_1^\infty$  approaches  $\infty$  and the sequence  $\{-n^2\}_1^\infty$  approaches  $-\infty$ .

(ii) Consider the sequence  $\{(-1)^{n+1}n\}_1^\infty$ . Although this is not a convergent sequence, it also does not approach to  $\infty$  or  $-\infty$ . Such sequences classified as

**Definition 2.2.13. Oscillating sequence:** If a sequence  $\{a_n\}$  does not converge to a value in  $\mathbb{R}$  and also does not diverge to  $\infty$  or  $-\infty$ , we say that  $\{a_n\}$  oscillates.

**Example 2.2.14.** The sequences  $\{(-1)^n\}_1^\infty$  and  $\{(-1)^n n\}_1^\infty$  are oscillating sequences.

**Theorem 2.2.15.** Let  $\{a_n\}$  and  $\{b_n\}$  be two sequences.

(i) If  $\{a_n\}$  and  $\{b_n\}$  both diverges to  $\infty$ , then the sequences  $\{a_n + b_n\}$  and  $\{a_n b_n\}$  also diverges to  $\infty$ .

(ii) If  $\{a_n\}$  diverges to  $\infty$  and  $\{b_n\}$  converges then  $\{a_n + b_n\}$  diverges to  $\infty$ .

The next example shows that the sequence formed by taking difference of two diverging sequences may converge.

**Problem 11.** Determine the convergence/divergence of the sequence  $\{\sqrt{n+1} - \sqrt{n}\}_{n=1}^\infty$ .

We know that  $\sqrt{n+1}$  and  $\sqrt{n}$  both diverges to  $\infty$ . But the sequence  $\{\sqrt{n+1} - \sqrt{n}\}_{n=1}^\infty$  converges to 0.

To see this, notice that, for a given  $\epsilon > 0$ ,

$$\sqrt{n+1} - \sqrt{n} < \epsilon \iff 1 < \epsilon^2 + 2\epsilon\sqrt{n}.$$

Thus, if  $N$  is such that  $N > \frac{1}{4\epsilon^2}$ , then for all  $n \geq N$ ,  $\sqrt{n+1} - \sqrt{n} < \epsilon$ . Thus  $\sqrt{n+1} - \sqrt{n}$  converges to 0.

### 2.3 Bolzano-Weierstrass theorem

Suppose a sequence is neither convergent nor approaches  $\pm\infty$ . Then we ask the question whether there is a subsequence that either converges or approaching  $\pm\infty$ . We discuss this in this section.

**Definition 2.3.1. Monotone sequence:**

A sequence  $\{a_n\}$  of real numbers is called a nondecreasing sequence if  $a_n \leq a_{n+1}$  for all  $n \in \mathbb{N}$  and  $\{a_n\}$  is called a nonincreasing sequence if  $a_n \geq a_{n+1}$  for all  $n \in \mathbb{N}$ . A sequence that is nondecreasing or nonincreasing is called a monotone sequence.

**Examples 2.3.2.**

- (i) The sequences  $\{1 - \frac{1}{n}\}$ ,  $\{n^3\}$  are nondecreasing sequences.
- (ii) The sequences  $\{\frac{1}{n}\}$ ,  $\{\frac{1}{n^2}\}$  are nonincreasing sequences.

**Remark 2.1.** A nondecreasing sequence which is not bounded above diverges to  $\infty$ . Similarly, a nonincreasing sequence which is not bounded below diverges to  $-\infty$ .

**Example 2.3.3.** If  $b > 1$ , then the sequence  $\{b^n\}_1^\infty$  diverges to  $\infty$ .

**Theorem 2.3.4.**

- (i) A nondecreasing sequence which is bounded above is convergent.
- (ii) A nonincreasing sequence which is bounded below is convergent.

*Proof.* (i) Let  $\{a_n\}$  be a nondecreasing, bounded above sequence and  $a = \sup_{n \in \mathbb{N}} a_n$ . Since the sequence is bounded,  $a \in \mathbb{R}$ . We claim that  $a$  is the limit point of the sequence  $\{a_n\}$ . Indeed, let  $\epsilon > 0$  be given. Since  $a - \epsilon$  is not an upper bound for  $\{a_n\}$ , there exists  $N \in \mathbb{N}$  such that  $a_N > a - \epsilon$ . As the sequence is nondecreasing, we have  $a - \epsilon < a_N \leq a_n$  for all  $n \geq N$ . Also it is clear that  $a_n \leq a$  for all  $n \in \mathbb{N}$ . Thus,

$$a - \epsilon \leq a_n \leq a + \epsilon, \quad \forall n \geq N.$$

The proof of (ii) is similar to (i) and is left as an exercise.

**Examples 2.3.5.**

- (i) If  $0 < b < 1$ , then the sequence  $\{b^n\}_1^\infty$  converges to 0.

**Solution.** We may write  $b^{n+1} = b^n b < b^n$ . Hence  $\{b^n\}$  is nonincreasing. Since  $b^n > 0$  for all  $n \in \mathbb{N}$ , the sequence  $\{b^n\}$  is bounded below. Hence, by the above theorem,  $\{b^n\}$  converges.

Let  $L = \lim_{n \rightarrow \infty} b^n$ . Then,

$$\lim_{n \rightarrow \infty} b^{n+1} = b \lim_{n \rightarrow \infty} b^n = bL.$$

Thus the sequence  $\{b^{n+1}\}$  converges to  $b \cdot L$ . On the other hand,  $\{b^{n+1}\}$  is a subsequence of  $\{b^n\}$ . Hence  $L = b \cdot L$  which implies  $L = 0$  as  $b \neq 1$ .

(ii) The sequence  $\left\{ \left(1 + \frac{1}{n}\right)^n \right\}_1^\infty$  is convergent.

**Solution.** Let  $a_n = \left(1 + \frac{1}{n}\right)^n = \sum_{k=0}^n \binom{n}{k} \left(\frac{1}{n}\right)^k$ . For  $k = 1, 2, \dots, n$ , the  $(k+1)^{th}$  term in the expansion is

$$\frac{n(n-1)(n-2)\cdots(n-k+1)}{1 \cdot 2 \cdots k} \frac{1}{n^k} = \frac{1}{k!} \left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) \cdots \left(1 - \frac{k-1}{n}\right). \quad (2.6)$$

Similarly, if we expand  $a_{n+1}$ , then we obtain  $(n+2)$  terms in the expansion and for  $k = 1, 2, 3, \dots$ , the  $(k+1)^{th}$  term is

$$\frac{1}{k!} \left(1 - \frac{1}{n+1}\right) \left(1 - \frac{2}{n+1}\right) \cdots \left(1 - \frac{k-1}{n+1}\right) < \frac{1}{k!}. \quad (2.7)$$

It is clear that left hand side of (2.7) is greater than or equal to right hand side of (2.6) and hence  $a_n \leq a_{n+1}$  which implies that  $\{a_n\}$  is nondecreasing. Further from (2.7),

$$a_n = \left(1 + \frac{1}{n}\right)^n = \sum_{k=0}^n \binom{n}{k} \left(\frac{1}{n}\right)^k < 1 + \sum_{k=1}^n \frac{1}{k!} < 1 + 1 + \sum_{k=2}^n \frac{1}{2^{k-1}} < 3, \forall n.$$

$$\left(k \geq 3 \implies k! > 2^{k-1}\right)$$

Thus  $\{a_n\}$  is a bounded monotone sequence and hence convergent.

**Theorem 2.3.6.** Every sequence has a monotone sub-sequence.

*Proof.* Pick  $x_{N_1}$  such that  $x_n \leq x_{N_1}$  for all  $n > N_1$ . We call such  $x_N$  as "peak". If we are able to pick infinitely many  $x'_{N_i}$ s, then  $\{x_{N_i}\}$  is decreasing and we are done. If there are only finitely many  $x'_N$ s and let  $x_{n_1}$  be the last peak. Then for  $n_2 > n_1$ ,  $x_{n_2}$  is not a peak. That means we can choose  $n_3$  such that  $x_{n_3} \geq x_{n_2}$ . Again  $x_{n_3}$  is not a peak. So we can choose  $x_{n_4}$  such that  $x_{n_4} \geq x_{n_3}$ . Proceeding this way, we get a non-decreasing sub-sequence  $\{x_{n_2}, x_{n_3}, x_{n_4}, \dots\}$ .

From the figure below 3

**Theorem 2.3.7. (Bolzano-Weierstrass)** Every bounded sequence has a convergent subsequence.

*Proof.* Proof is a consequence of theorem 2.3.6.

**Theorem 2.3.8. Nested Interval theorem:** Let  $I_n = [a_n, b_n], n \geq 1$  be non-empty closed, bounded intervals such that

$$I_1 \supset I_2 \supset I_3 \cdots \supset I_n \supset I_{n+1} \cdots$$

and  $\lim_{n \rightarrow \infty} (b_n - a_n) = 0$ . Then  $\cap_{n=1}^\infty I_n$  contains precisely one point

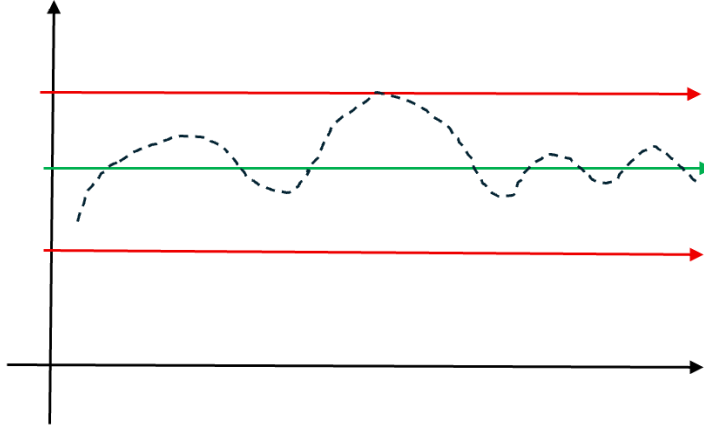


Figure 2: Bolzano-Weierstrass

*Proof.* Since  $\{a_n\}, \{b_n\} \subset [a_1, b_1]$ ,  $\{a_n\}, \{b_n\}$  are bounded sequences. By Bolzano-Weierstrass theorem, there exists sub sequences  $a_{n_k}, b_{n_k}$  and  $a, b$  such that  $a_{n_k} \rightarrow a, b_{n_k} \rightarrow b$ . Since  $a_n$  is increasing  $a_1 < a_2 < \dots \leq a$  and  $b_1 > b_2 > \dots \geq b$ . It is easy to see that  $a \leq b$ . Also since  $0 = \lim a_n - b_n = a - b$ , we have  $a = b$ . It is again easy to show that there is no other point in  $\cap_{n=1}^{\infty} I_n$ .

**Remark 2.2.** *closedness of  $I_n$  cannot be dropped. for example the sequence  $\{(0, \frac{1}{n})\}$ . Then  $\cap_{n=1}^{\infty} (0, \frac{1}{n}) = \emptyset$  because there cannot be any element  $x$  such that  $0 < x < \frac{1}{n}$  else Archimedean property fails.*

Earlier we proved that there is a bijection from the set of natural numbers to the set of rationals. In other words, the set of rationals is countable. Below we show that this is not true for the set of real numbers.

**Corollary 2.3.9.**  $\mathbb{R}$  is uncountable.

*Proof.* It is enough to show that  $[0, 1]$  is uncountable. If not, there exists an onto map  $f : \mathbb{N} \rightarrow [0, 1]$ . Now subdivide  $[0, 1]$  into 3 equal parts so that choose  $J_1$  such that  $f(1) \notin J_1$ . Now subdivide  $J_1$  into 3 equal parts and choose  $J_2$  so that  $f(2) \notin J_2$ . Continue the process to obtain  $J_n$  so that  $f(n) \notin J_n$ . These  $J_n$  satisfy the hypothesis of above theorem, so  $\cap_{n=1}^{\infty} J_n = \{x\}$  and  $x \in [0, 1]$ . By the construction, there is no  $n \in \mathbb{N}$  such that  $f(n) = x$ . contradiction to  $f$  is onto.

## 2.4 Cauchy sequences

**Definition 2.4.1.** A sequence  $\{a_n\}$  is called a Cauchy sequence if for any given  $\epsilon > 0$ , there exists  $N \in \mathbb{N}$  such that

$$|a_n - a_m| < \epsilon, \text{ for all } n, m \geq N$$

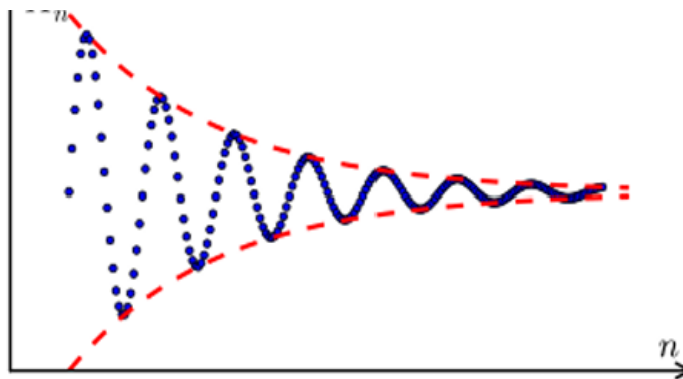


Figure 3: Cauchy sequence

**Theorem 2.4.2.** *If  $\{a_n\}$  is a convergent sequence then  $\{a_n\}$  is a Cauchy sequence*

*Proof.* Let  $\{a_n\}$  be a sequence that converges to  $L$  (say). Then for any  $\epsilon > 0$ , there exists  $N \in \mathbb{N}$  such that

$$|a_n - L| < \frac{\epsilon}{2}, \text{ for all } n \geq N.$$

Thus if  $n, m \geq N$ , we have

$$|a_n - a_m| \leq |a_n - L| + |a_m - L| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus  $\{a_n\}$  is a Cauchy sequence.

**Lemma 2.4.3.** *If  $\{a_n\}$  is a Cauchy sequence, then  $\{a_n\}$  is bounded.*

*Proof.* Since  $\{a_n\}$  is a Cauchy sequence, for  $\epsilon = 1$  there exists  $N \in \mathbb{N}$  such that

$$|a_n - a_m| < 1, \forall n, m \geq N.$$

In particular,

$$|a_n - a_N| < 1, \forall n \geq N.$$

Hence if  $n \geq N$ , then

$$|a_n| \leq |a_n - a_N| + |a_N| < 1 + |a_N|, \forall n \geq N.$$

Let  $M = \max\{|a_1|, |a_2|, \dots, |a_{N-1}|, 1 + |a_N|\}$ . Then  $|a_n| \leq M$  for all  $n \in \mathbb{N}$ . Hence  $\{a_n\}$  is bounded.

**Theorem 2.4.4.** *If  $\{a_n\}$  is a Cauchy sequence, then  $\{a_n\}$  is convergent.*

*Proof.* Since  $\{a_n\}$  is a Cauchy sequence. By Bolzano-Weierstrass theorem there exists a convergent sub-sequence  $\{a_{n_k}\}$  which converges to  $L$  (say). Now we claim that the sequence  $\{a_n\}$

itself converges to  $L$ . Let  $\epsilon > 0$ . Choose  $N_1, N_2$  such that

$$n, n_k \geq N_1 \implies |a_n - a_{n_k}| < \epsilon/2$$

$$n_k \geq N_2 \implies |a_{n_k} - a| < \epsilon/2.$$

Then by taking  $N = \max\{N_1, N_2\}$ , we get

$$n, n_k \geq N \implies |a_n - a| \leq |a_n - a_{n_k}| + |a_{n_k} - a| < \epsilon.$$

Hence the claim.

Therefore, we proved the following theorem

**Theorem 2.4.5.** *A sequence  $\{a_n\}$  converges if and only if  $\{a_n\}$  is a Cauchy sequence.*

**Problem 12.** *Let  $\{a_n\}$  be defined as  $a_1 = 1, a_{n+1} = 1 + \frac{1}{a_n}$ . The show that  $\{a_n\}$  is Cauchy.*

**Solution:** Note that  $a_n > 1$  and  $a_n a_{n-1} = a_{n-1} + 1 > 2$ . Then

$$|a_{n+1} - a_n| = \left| \frac{a_{n-1} - a_n}{a_n a_{n-1}} \right| \leq \frac{1}{2} |a_n - a_{n-1}| \leq \frac{1}{2^{n-1}} |a_2 - a_1|, \forall n \geq 2.$$

Hence

$$\begin{aligned} |a_m - a_n| &\leq |a_m - a_{m-1}| + |a_{m-1} - a_{m-2}| + \dots + |a_{n+1} - a_n| \\ &\leq |a_2 - a_1| \left( \frac{1}{2} + \frac{1}{2^2} + \dots \right) \\ &= \frac{4}{2^n} |a_2 - a_1| = \frac{4}{2^n} \end{aligned}$$

So given,  $\epsilon > 0$ , we can choose  $N$  such that  $\frac{4}{2^N} < \frac{\epsilon}{2}$ .

Indeed the following holds,

**Theorem 2.4.6.** *Let  $\{a_n\}$  be a sequence such that  $|a_{n+1} - a_n| < \alpha |a_n - a_{n-1}|$  for all  $n \geq N$  for some  $N$  and  $0 < \alpha < 1$ . Then  $\{a_n\}$  is a Cauchy sequence.*

*Proof.* Proof follows as in the previous example.

In the above theorem if  $\alpha = 1$ , then we cannot say if the sequence is Cauchy or Not. For example

**Example 2.4.7.** Let  $a_n = \sum_{k=1}^n \frac{1}{k}$ . Then it is easy to see that

$$\frac{|a_{n+1} - a_n|}{|a_n - a_{n-1}|} = \frac{n}{n+1} < 1.$$

But the sequence  $\{a_n\}$  diverges. Indeed,  $a_{2^n} > 1 + \frac{n}{2}$ , by induction.

On the other hand if we choose  $a_n = \sum_{k=1}^n \frac{1}{k^2}$ . Then

$$\frac{|a_{n+1} - a_n|}{|a_n - a_{n-1}|} = \frac{n^2}{(n+1)^2} < 1.$$

But the sequence  $\{a_n\}$  converges (This will be proved in the next section while studying infinite series).

## 2.5 Limit superior and limit inferior

In this section we define the limit superior and limit inferior of a sequence  $\{a_n\}$ . For simplicity, we take bounded sequence  $\{a_n\}$ . However the same definition also holds for unbounded sequences.

**Definition 2.5.1.** The limit superior of the sequence  $\{a_n\}$ , denoted by  $\limsup_{n \rightarrow \infty} a_n$ , is defined as

$$\limsup_{n \rightarrow \infty} a_n := \inf_{k \in \mathbb{N}} \sup_{n \geq k} a_n.$$

Similarly limit inferior of the sequence  $\{a_n\}$ , denoted by  $\liminf_{n \rightarrow \infty} a_n$ , is defined as

$$\liminf_{n \rightarrow \infty} a_n := \sup_{k \in \mathbb{N}} \inf_{n \geq k} a_n.$$

**Examples 2.5.2.** 1. Consider the sequence  $\{a_n\} = \{(-1)^{n+1}\}$ .

Then  $a_n = \{1, -1, 1, -1, \dots\}$ . Therefore  $\beta_n = \sup\{a_m, m \geq n\} = 1$  and  $\alpha_n = \inf\{a_m, m \geq n\} = -1$ . Therefore,  $\liminf a_n = -1, \limsup a_n = 1$ .

2. Consider the sequence  $\{a_n\} = \{\sin \frac{n\pi}{3}\}$ .

Then by listing the elements of  $\{a_n\}$ , it is easy to observe that each element of the sequence is repeated infinitely many times. So  $\beta_n = \sup\{a_m, m \geq n\} = \frac{\sqrt{3}}{2}$ . Therefore  $\limsup a_n = \frac{\sqrt{3}}{2}$ . Similarly one can see  $\liminf a_n = -\frac{\sqrt{3}}{2}$ .

**Lemma 2.5.3.**

(i) If  $\{a_n\}$  is a bounded sequence, then  $\limsup_{n \rightarrow \infty} a_n \geq \liminf_{n \rightarrow \infty} a_n$ .

(ii) If  $\{a_n\}$  and  $\{b_n\}$  are bounded sequences of real numbers and if  $a_n \leq b_n$  for all  $n \in \mathbb{N}$ , then

$$\limsup_{n \rightarrow \infty} a_n \leq \limsup_{n \rightarrow \infty} b_n$$

and

$$\liminf_{n \rightarrow \infty} a_n \leq \liminf_{n \rightarrow \infty} b_n.$$

(iii) Let  $\{a_n\}$  and  $\{b_n\}$  are bounded sequences of real numbers. Then

$$\limsup_{n \rightarrow \infty} (a_n + b_n) \leq \limsup_{n \rightarrow \infty} a_n + \limsup_{n \rightarrow \infty} b_n$$

and

$$\liminf_{n \rightarrow \infty} (a_n + b_n) \geq \liminf_{n \rightarrow \infty} a_n + \liminf_{n \rightarrow \infty} b_n.$$

**Example 2.5.4.** Consider the sequences  $\{(-1)^n\}$  and  $\{(-1)^{n+1}\}$ . Here  $a_n = (-1)^n$  and  $b_n = (-1)^{n+1}$ . Also  $\limsup_{n \rightarrow \infty} a_n = \limsup_{n \rightarrow \infty} b_n = 1$ . But  $a_n + b_n = 0$  for all  $n \in \mathbb{N}$  and hence  $\limsup_{n \rightarrow \infty} (a_n + b_n) = 0$ . Thus a strict inequality may hold in (iii) the above Lemma.

**Theorem 2.5.5.** If  $\{a_n\}$  is a bounded sequence, then there exist sub-sequences  $\{a_{n_k}\}$  and  $\{b_{n_k}\}$  such that

$$\limsup a_n = \lim a_{n_k} \quad \text{and} \quad \liminf a_n = \lim b_{n_k}.$$

*Proof.* Since  $\{a_n\}$  is bounded,  $\limsup a_n = \alpha$  exists. Then from the definition, for each  $k$  large enough

$$\alpha < \sup_{n \geq k} a_n < \alpha + \frac{1}{k}$$

i.e., there exists  $a_{n_k}$  such that

$$\alpha - \frac{1}{k} < a_{n_k} < \alpha + \frac{1}{k}.$$

Therefore,  $a_{n_k} \rightarrow \alpha$  as  $k \rightarrow \infty$ . Similarly, one can obtain  $b_{n_k}$ .

Now the Bolzano-Weierstrass theorem is an easy consequence.

**Theorem 2.5.6.** If there exists a subsequence  $a_{n_k} \rightarrow t$ . Then  $t \leq s := \limsup a_n$ .

*Proof.* Suppose NOT. Then choose  $\epsilon > 0$  such that  $t - \epsilon > s$ . Then we can find  $N$  such that

$$n \geq N \implies a_n < t - \epsilon$$

Therefore  $|a_n - t| > \epsilon$  for all  $n \geq N$ . Hence such a sequence cannot have a convergent subsequence.

**limsup and liminf of Unbounded sequences:** In case of unbounded sequences, either  $\limsup$  or  $\liminf$  or both can approach  $\infty$ . Even in this case, one can show the existence of subsequences that approach infinity. In this case either  $\limsup$  or  $\liminf$  or both of them are not finite.

**Remark 2.3.** If we can find the limits (finite or approaching  $\pm\infty$ ) of all subsequences of  $\{a_n\}$ . Then  $\limsup$  is nothing but the supremum of all these limits. Similarly,  $\liminf$  is the infimum of all these limits.

**Problem 13.** Find  $\limsup$  and  $\liminf$  of the sequence  $\{a_n\} = \{\frac{1}{2}, \frac{2}{3}, \frac{1}{3}, \frac{3}{4}, \dots\}$ .

First we note that this is a bounded sequence. By observation we see that there are two convergent sub-sequences  $\{\frac{n-1}{n}\}$  and  $\{\frac{1}{n}\}$ . Hence we see that  $\limsup a_n = 1$  and  $\liminf a_n = 0$ .

**Problem 14.** Find  $\limsup$  and  $\liminf$  of  $\{a_n\}_{n=1}^{\infty}$  where  $a_n = (1 + (-1)^n + \frac{1}{2^n})^{\frac{1}{n}}$ .

The sequence  $\{a_n\}$  is bounded and has two convergent subsequences  $\{\frac{1}{2}\}$  and  $\{(2 + \frac{1}{2^n})^{\frac{1}{n}}\}$ . So the two limits are  $\frac{1}{2}$  and 1. Therefore,  $\limsup a_n = 1$  and  $\liminf a_n = \frac{1}{2}$ .

**Problem 15.** Find  $\limsup$  and  $\liminf$  of the sequence

$$\left\{1, \frac{1}{2}, 2, \frac{1}{3}, 3, \frac{1}{4}, 4, \dots\right\}$$

Here we see that there is one sub-sequence  $\{\frac{1}{n}\}$  that converges to 0 and another sub-sequence  $\{n\}$  that approaches  $\infty$ . So the  $\limsup$  is  $\infty$  and  $\liminf$  is 0.

**Problem 16.** Find  $\limsup$  and  $\liminf$  of the sequence  $\{2 + (-1)^n\}$

Here the given sequence  $\{a_n\}$  may be listed as

$$\{1, 3, 1, 3, 1, \dots\}$$

We observe that the elements 1 and 3 repeatedly appear in the sequence. So  $\beta_n = \sup_{k \geq n} a_k = 3$  and  $\alpha_n = \inf_{k \geq n} a_k = 1$ . Hence

$$\limsup a_n = 3, \quad \liminf a_n = 1.$$

**Theorem 2.5.7.** If  $\{a_n\}$  is a convergent sequence, then

$$\liminf_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} a_n = \limsup_{n \rightarrow \infty} a_n.$$

*Proof.* Let  $L = \lim_{n \rightarrow \infty} a_n$ . Then given  $\epsilon > 0$  there exists  $N \in \mathbb{N}$  such that

$$|a_n - L| < \epsilon, \quad \forall n \geq N.$$

Equivalently  $L - \epsilon < a_n < L + \epsilon$ , for all  $n \geq N$ . Thus, if  $n \geq N$ ,  $L + \epsilon$  is an upper bound for the set  $\{a_k | k \geq N\}$ . If  $\alpha_k := \sup\{a_k | k \geq n\}$ , then we note that  $L - \epsilon < \alpha_N \leq L + \epsilon$  and  $\alpha_{N+1} < L + \epsilon, \dots, \alpha_n < L + \epsilon$  for all  $n \geq N$  (As  $\alpha_n$  is decreasing, if all  $\alpha_n = L + \epsilon$  then limit cannot be  $L$  (think!)). Also  $a_n > L - \epsilon, n \geq N \implies \alpha_n \geq L - \epsilon, n \geq N$ . Therefore,  $\lim \alpha_n = L$ . Hence  $\limsup a_n = L$ . Similarly, one can prove the  $\liminf a_n = L$ .

**Theorem 2.5.8.** If  $\{a_n\}$  is a bounded sequence and if  $\limsup_{n \rightarrow \infty} a_n = \liminf_{n \rightarrow \infty} a_n = L, L \in \mathbb{R}$ , then  $\{a_n\}$  is a convergent sequence.

*Proof.* Notice that

$$\limsup_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} (\sup\{a_k | k \geq n\})$$

and

$$\liminf_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} (\inf\{a_k | k \geq n\}).$$

Given that  $L = \limsup_{n \rightarrow \infty} a_n$ . Thus for  $\epsilon > 0$ , there exists  $N_1 \in \mathbb{N}$  such that

$$|\sup\{a_n, a_{n+1}, \dots\} - L| < \epsilon, \forall n \geq N_1.$$

This implies

$$a_n < L + \epsilon, \forall n \geq N_1 \quad (2.8)$$

Similarly there exists  $N_2 \in \mathbb{N}$  such that

$$|\inf\{a_n, a_{n+1}, \dots\} - L| < \epsilon, \forall n \geq N_2.$$

This implies

$$L - \epsilon < a_n, \forall n \geq N_2 \quad (2.9)$$

Let  $N = \max\{N_1, N_2\}$ . Then from (2.8) and (2.9) we get

$$|a_n - L| < \epsilon, \forall n \geq N.$$

Thus the sequence  $\{a_n\}$  converges.

**Problem 17.** Find  $\limsup$  and  $\liminf$  of  $\{a_n\}$ , where  $a_n = \begin{cases} 2^{\frac{1}{n}} & n \text{ odd} \\ 2^{-\frac{1}{n}} & n \text{ even} \end{cases}$

Here we see that there are two subsequences of which one is  $\{2^{\frac{1}{n}}\}$  and other one is  $\{2^{-\frac{1}{n}}\}$ . Both of them converge to 1. So  $\limsup$  and  $\liminf$  are equal to the limit of the sequence. Finally we have the following more precise version of theorem 1.6.6

**Theorem 2.5.9.** Let  $\{a_n\}$  be any sequence of positive real numbers. Then we have

$$\liminf \left| \frac{a_{n+1}}{a_n} \right| \leq \liminf |a_n|^{\frac{1}{n}} \leq \limsup |a_n|^{\frac{1}{n}} \leq \limsup \left| \frac{a_{n+1}}{a_n} \right|.$$

*Proof.* The inequality in the middle is trivial. Now we show the right end inequality. Let  $L = \limsup \left| \frac{a_{n+1}}{a_n} \right|$ . W.l.g assume  $L < \infty$ . Let  $\epsilon > 0$ . Then there exists  $N \in \mathbb{N}$  such that

$$\left| \frac{a_{n+1}}{a_n} \right| < L + \epsilon \quad \forall n \geq N.$$

Then for any  $n > N$ , we can write

$$|a_n| = \left| \frac{a_n}{a_{n-1}} \right| \left| \frac{a_{n-1}}{a_{n-2}} \right| \dots \left| \frac{a_{N+1}}{a_N} \right| |a_N| < (L + \epsilon)^{n-N} |a_N| = (L + \epsilon)^n ((L + \epsilon)^{-N} |a_N|).$$

Now taking  $a = ((L + \epsilon)^{-N} |a_N|)$ , we have,  $|a_n|^{1/n} < (L + \epsilon) a^{1/n}$  for  $n > N$ . Since  $\lim_{n \rightarrow \infty} a^{1/n} = 1$ ,

we conclude that  $\limsup |a_n|^{1/n} < (L + 2\epsilon)$ . Since  $\epsilon$  is arbitrary, we get the result. Similarly, we can prove the first inequality.

The following example shows that the inequality in the above theorem can be strict;

**Example 2.5.10.** Let  $a_n = \begin{cases} 2^n & n \text{ is even} \\ 2^{n-1} & n \text{ is odd} \end{cases}$ . Then

$$\left| \frac{a_{n+1}}{a_n} \right| = \begin{cases} \frac{2^{(n+1)-1}}{2^n} & n \text{ even} \\ \frac{2^{n+1}}{2^{n-1}} & n \text{ odd} \end{cases} = \begin{cases} 1 & n \text{ even} \\ 4 & n \text{ odd} \end{cases}$$

$$\therefore \limsup \left| \frac{a_{n+1}}{a_n} \right| = 4 \text{ and } \liminf \left| \frac{a_{n+1}}{a_n} \right| = 1$$

where as

$$\limsup |a_n|^{1/n} = \lim |a_n|^{1/n} = 2.$$

**Problem 18.** Find  $\limsup$  and  $\liminf$  of  $|a_n|^{1/n}$  and  $\left| \frac{a_{n+1}}{a_n} \right|$  for the sequence defined by

$$a_n = \left( 1 + \sin \frac{n\pi}{3} \right)^n$$

Hint: All 4 of the them are different numbers.

**Problem 19.** Suppose  $a_n > 0$  for all  $n \in \mathbb{N}$ . Then

1.  $\limsup \frac{a_{n+1}}{a_n} = l < 1$ , then show that  $a_n \rightarrow 0$  as  $n \rightarrow \infty$ .
2.  $\liminf \frac{a_{n+1}}{a_n} = l > 1$ , then show that  $\{a_n\}$  is an unbounded sequence.

By the above theorem 2.5.9,  $\limsup a_n^{1/n} \leq l < 1$ . Then the definition of  $\limsup$ , we can find  $\epsilon > 0$  and  $N_0$  such that  $l + \epsilon < 1$  and

$$a_n^{1/n} < l + \epsilon < 1, \forall n \geq N_0$$

This implies  $a_n \rightarrow 0$  as  $n \rightarrow \infty$ . Therefore (1) follows.

similarly, (2) follows from the definition of  $\liminf$  and the above theorem 2.5.9.

**Problem 20.** Show that the sequence  $\left\{ \left( \sin \frac{n\pi}{3} \right)^n \right\}$  converges to 0.

In this case  $a_n^{\frac{1}{n}} = \sin \frac{n\pi}{3}$ . Therefore  $\limsup a_n^{\frac{1}{n}} = \frac{\sqrt{3}}{2} < 1$ . Therefore by the above theorem  $a_n \rightarrow 0$ .

### 3 Infinite Series

In this section we study the sum of infinite many numbers. We are familiar with geometric series whose sum is finite if the common ratio is strictly less than 1. The idea of infinite series is seen in the decimal expansion of irrational numbers like  $\pi$  and  $e$ , where the decimal expansion is not terminating one. We would like to study such infinite series and their properties.

#### 3.1 Definitions

**Definition 3.1.1.** Let  $\{a_n\}$  be a sequence of real numbers.

- a) An expression of the form  $a_1 + a_2 + \dots + a_n + \dots$  is called an infinite series.
- b) The sequence  $\{s_n\}$ , defined by  $s_n = \sum_{k=1}^n a_k$ , is called the sequence of partial sums of the series.

**Definition 3.1.2.** If the sequence of partial sums converges to a limit  $L$ , we say that the series converges and its sum is  $L$ . If the sequence of partial sums does not converge, we say that the series diverges.

#### Examples 3.1.3.

- 1) If  $0 < x < 1$ , then  $\sum_{n=0}^{\infty} x^n$  converges to  $\frac{1}{1-x}$ .

**Solution.** Let us consider the sequence of partial sums  $\{s_n\}$ , where  $s_n = \sum_{k=0}^n x^k$ . Here

$$s_n = \sum_{k=0}^n x^k = \frac{1 - x^{n+1}}{1 - x} = \frac{1}{1 - x} - \frac{x^{n+1}}{1 - x}, \quad n \in \mathbb{N}.$$

As,  $0 < x < 1$ ,  $x^{n+1} \rightarrow 0$  as  $n \rightarrow \infty$ . Hence  $s_n \rightarrow \frac{1}{1-x}$ . Thus  $\sum x^n = \frac{1}{1-x}$ .

- 2) The series  $\sum_{n=1}^{\infty} \frac{1}{n}$  diverges.

**Solution.** Consider the sequence of partial sums  $\{s_n\}$ , where  $s_n = \sum_{k=1}^n \frac{1}{k}$ . Now, let us examine the subsequence  $s_{2n}$  of  $\{s_n\}$ . Here

$$s_2 = 1 + 1/2 = 3/2, \quad s_4 = s_2 + 1/3 + 1/4.$$

Now observe that  $s_4 - s_2 > \frac{1}{2}$  and so

$$s_{2n} - s_n = \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{2n} > \frac{1}{2n}n = \frac{1}{2}$$

Hence  $\{s_n\}$  cannot be a Cauchy sequence and hence cannot converge.

3) (Telescopic series:) Show that the series  $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$  converges to 1.

**Solution.** Consider the sequence of partial sums  $\{s_n\}$ . Then

$$s_n = \sum_{k=1}^n \frac{1}{k(k+1)} = \sum_{k=1}^n \left( \frac{1}{k} - \frac{1}{k+1} \right) = 1 - \frac{1}{n+1} \rightarrow 1.$$

Summarizing this observation, one has the following theorem on **Telescopic series**

**Theorem 3.1.4.** Suppose  $\{a_n\}$  is a sequence of real numbers such that  $a_n \rightarrow L$ . Then the series  $\sum (a_n - a_{n+1})$  converges to  $a_1 - L$ .

The following theorem is an easy consequence of the definition

**Theorem 3.1.5. Algebra of convergent series:**

- 1) If  $\sum_{n=1}^{\infty} a_n$  converges to  $L$  and  $\sum_{n=1}^{\infty} b_n$  converges to  $M$ , then the series  $\sum_{n=1}^{\infty} (a_n + b_n)$  converges to  $L + M$ .
- 2) If  $\sum_{n=1}^{\infty} a_n$  converges to  $L$  and if  $c \in \mathbb{R}$ , then the series  $\sum_{n=1}^{\infty} ca_n$  converges to  $cL$ .

Next we have a theorem that is a necessary condition for convergence of series.

**Theorem 3.1.6.** If  $\sum_{n=1}^{\infty} a_n$  converges, then  $\lim_{n \rightarrow \infty} a_n = 0$ .

*Proof.* Suppose  $\sum_{n=1}^{\infty} a_n = L$ . Then the sequence of partial sums  $\{s_n\}$  also converges to  $L$ . Now

$$a_n = s_n - s_{n-1} \rightarrow L - L = 0.$$

**Example 3.1.7.** If  $x > 1$ , then the series  $\sum_{n=1}^{\infty} x^n$  diverges.

We know that if  $x > 1$  then  $x^n$  diverges to  $\infty$ . Therefore by above theorem, the series  $\sum_{n=1}^{\infty} x^n$  diverges.

**Problem 21.** Show that the series  $\sum_{n=1}^{\infty} \frac{n}{n+1}$  diverges.

Here  $a_n = \frac{n}{n+1} \rightarrow 1$  as  $n \rightarrow \infty$ . So by the theorem 3.1.6 the series diverges.

The above theorem 3.1.6 is ONLY necessary but NOT a sufficient condition, for example

**Example 3.1.8.** Consider the series  $\sum_{n=1}^{\infty} \frac{1}{n}$ . Here  $a_n = \frac{1}{n} \rightarrow 0$ , but the series diverges.

### 3.2 Tests for convergence

As a first result we have the following comparison test:

**Theorem 3.2.1.** *Let  $\{a_n\}, \{b_n\}$  be sequences of positive reals such that  $a_n \leq b_n$ . If  $\sum b_n$  converges then  $\sum a_n$  converges. Also, if  $\sum a_n$  diverges then  $\sum b_n$  diverges.*

*Proof.* Let  $s_n = a_1 + a_2 + \dots + a_n$  and  $t_n = b_1 + b_2 + \dots + b_n$  be the partial sum of  $\sum a_n, \sum b_n$  respectively. Then  $s_n \leq t_n$ . Since  $\sum b_n$  converges, we have  $\{t_n\}$  converges and is bounded. Now since  $\{s_n\}$  is monotonically increasing sequence that is bounded above, we get the convergence of  $\{s_n\}$  and hence the convergence of  $\sum a_n$ .

If  $\sum a_n$  diverges then  $s_n \rightarrow \infty$ . Then  $t_n \geq s_n$  implies  $t_n$  diverges to infinity.

**Problem 22.** *Test the convergence of*

$$(a) \sum \frac{1}{2^n + n} \quad (b) \sum \frac{n}{n^2 - \sin^2 n}$$

For (a), note that  $2^n + n > 2^n$  and  $\sum \frac{1}{2^n}$  converges. For (b) note that  $n^2 - \sin^2 n < n^2$  and the series  $\sum \frac{1}{n}$  diverges.

**Problem 23.** *Show that the series  $\sum_{n=0}^{\infty} \frac{1}{n!}$  is convergent and the limit is not a rational number. This number is denoted with  $e$ .*

**Solution:** We can see that

$$n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdots n > 1 \cdot 2 \cdot 2 \cdots 2 = 2^{n-1}$$

Hence by comparison test we can take  $b_n = \sum \frac{1}{2^{n-1}}$ , a convergent geometric series. Hence the given series converges.

To show that the limit is not a rational number. Let  $e$  be the limit. If  $e$  is a rational number, then

$$\frac{a}{b} = e = \sum_{n=0}^{\infty} \frac{1}{n!}$$

for some  $a$  and  $b$ , natural numbers. Then multiplying by  $b!$

$$a(b-1)! = \sum_{n=0}^{\infty} \frac{b!}{n!} = N + \frac{1}{b+1} + \frac{1}{(b+1)(b+2)} + \cdots \quad (3.10)$$

for some natural number  $N$ . Then the right hand side can be estimated from below and above as

$$N < N + \frac{1}{b+1} + \frac{1}{(b+1)(b+2)} + \cdots < N + \sum_{k=1}^{\infty} \frac{1}{(b+1)^k} = N + \frac{1}{b} < N + 1$$

Therefore the right hand side of (3.10) is not a natural number. But left hand side  $a(b-1)!$  is a natural number. Both can not be true together. Hence the proof.

**Problem 24.** Show that

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = \sum_{n=0}^{\infty} \frac{1}{n!} := e$$

**Solution:** We know that both the limits exists. Let

$$s = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n, \text{ and } e = \sum_{n=0}^{\infty} \frac{1}{n!}.$$

From the binomial expansion

$$\left(1 + \frac{1}{n}\right)^n = \sum_{k=0}^n \frac{n!}{k!(n-k)!} \frac{1}{n^k} = \sum_{k=0}^n \frac{1}{k!} \left(1 - \frac{1}{n}\right) \cdots \left(1 - \frac{k+1}{n}\right) \leq \sum_{k=0}^n \frac{1}{k!}$$

This implies  $s \leq e$ .

Also we see that for any fixed  $p$ ,

$$\left(1 + \frac{1}{n}\right)^n \geq \sum_{k=0}^p \frac{1}{k!} \left(1 - \frac{1}{n}\right) \cdots \left(1 - \frac{k+1}{n}\right)$$

and we see that right hand side converges (increases) to  $\sum_{k=0}^{\infty} \frac{1}{k!}$ . So we may write

$$-\epsilon + \sum_{k=0}^p \frac{1}{k!} \leq \left(1 + \frac{1}{n}\right)^n$$

where  $\epsilon$  converges to 0 as  $n \rightarrow \infty$ . This implies

$$\sum_{k=0}^p \frac{1}{k!} \leq s$$

So taking limit  $p \rightarrow \infty$ , we get  $e \leq s$ .

**Theorem 3.2.2. Cauchy condensation test:**

Let  $\{a_n\}_1^{\infty}$  be an decreasing sequence of non-negative real numbers. Then  $\sum_{n=1}^{\infty} a_n$  converges if and only if  $\sum_{n=0}^{\infty} 2^n a_{2^n}$  converges.

*Proof.* Let  $s_n$  and  $t_n$  be the sequence of partial sums of  $\sum a_n$  and  $\sum 2^n a_{2^n}$  respectively. Then  $s_n$  and  $t_n$  are monotonically increasing sequences. We know that such sequences converge if

they are bounded from above. proof follows from the observation that

$$\begin{aligned}
 s_{2^n} &= \sum_{k=1}^{2^n} a_k = a_1 + a_2 + (a_3 + a_4) + (a_5 + a_6 + a_7 + a_8) + \dots + (a_{2^{n-1}+1} + \dots + a_{2^n}) \\
 &\geq a_1 + a_2 + 2a_4 + 4a_8 + 8a_{16} + \dots 2^{n-1}a_{2^n} \\
 &= a_1 + \frac{1}{2}t_n.
 \end{aligned} \tag{3.11}$$

Therefore, if  $\{s_n\}$  converges then  $\{s_{2^n}\}$  converges and hence bounded from above. Now convergence of  $\{t_n\}$  follows from 3.11,  $\{t_n\}$ .

On the other hand,

$$\begin{aligned}
 s_{2^{n-1}} &= a_1 + (a_2 + a_3) + (a_4 + a_5 + a_6 + a_7) + (a_8 + \dots a_{15}) + (a_{2^{n-1}} + \dots + a_{2^n-1}) \\
 &\leq a_1 + 2a_2 + 4a_4 + 8a_8 + \dots + 2^{n-1}a_{2^{n-1}} = a_1 + t_{n-1}
 \end{aligned}$$

So if  $\{t_n\}$  converges, then  $\{s_{2^{n-1}}\}$  converges. Now the conclusion follows from  $s_n \leq s_{2^{n+1}-1}$  and the fact that  $\{s_n\}$  is monotonically increasing sequence.

### Examples 3.2.3.

- 1) Consider the series  $\sum_{n=1}^{\infty} \frac{1}{n^p}$ ,  $p > 0$ . Then, we have  $\sum_{n=1}^{\infty} 2^n \frac{1}{(2^n)^p} = \sum_{n=1}^{\infty} \frac{1}{(2^n)^{p-1}}$  which converges for  $p > 1$  and diverges for  $p \leq 1$ .
- 2) Consider the series  $\sum_{n=2}^{\infty} \frac{1}{n \log n}$ . Here  $\sum_{n=2}^{\infty} 2^n \frac{1}{2^n \log 2^n} = \frac{1}{\log 2} \sum_{n=2}^{\infty} \frac{1}{n}$  which diverges. Hence the given series diverges.

### Definition 3.2.4. Absolute convergence

For a series  $\sum_{n=1}^{\infty} a_n$  of real numbers, if  $\sum_{n=1}^{\infty} |a_n|$  converges then we say that  $\sum_{n=1}^{\infty} a_n$  converges absolutely.

### Examples 3.2.5.

- 1) The series  $\sum_{n=1}^{\infty} \frac{(-1)^n}{n!}$  converges absolutely.
- 2) The series  $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$  converges absolutely.

**Theorem 3.2.6.** If  $\sum_{n=1}^{\infty} a_n$  converges absolutely, then  $\sum_{n=1}^{\infty} a_n$  converges.

*Proof.* Let  $t_n = \sum_{k=1}^n |a_k|$ . As the series converges absolutely, the sequence  $\{t_n\}_1^{\infty}$  is Cauchy. Thus, given  $\epsilon > 0$ , there exists  $N \in \mathbb{N}$  such that

$$|t_m - t_n| < \epsilon \quad \forall m, n \geq N.$$

Let  $m > n$ . Then

$$|s_m - s_n| = \left| \sum_{i=n+1}^m a_i \right| \leq \sum_{i=n+1}^m |a_i| = |t_m - t_n| < \epsilon.$$

Thus the sequence  $\{s_n\}_1^\infty$  is Cauchy and hence converges. Thus  $\sum_1^\infty a_n$  converges.

**Theorem 3.2.7.** Let  $\sum_1^\infty a_n$  be a series of real numbers. Let  $p_n = \max\{a_n, 0\}$  and  $q_n = \min\{a_n, 0\}$ .

- a) If  $\sum a_n$  converges absolutely, then both  $\sum p_n$  and  $\sum q_n$  converges.
- b) If  $\sum a_n$  diverges then one of the  $\sum p_n$  or  $\sum q_n$  diverges.

*Proof.* Proof of (a) follows by the observation that  $p_n = (a_n + |a_n|)/2$  and  $q_n = (a_n - |a_n|)/2$ . Thus the convergence of the two series follows from the hypothesis. (b) is easy to prove.

### 3.3 Tests for absolute convergence

**Theorem 3.3.1** (Comparison test). Let  $\sum a_n$  be a series of real numbers. Then,  $\sum a_n$  converges absolutely if there is an absolutely convergent series  $\sum c_n$  with  $|a_n| \leq |c_n|$  for all  $n \geq N, N \in \mathbb{N}$ .

**Example 3.3.2.** The series  $\sum_{n=1}^\infty \frac{(-1)^n}{n!}$  converges absolutely because  $\left| \frac{(-1)^n}{n!} \right| = \left| \frac{1}{n!} \right| \leq \frac{1}{2^{n-1}}$  for  $n \geq 3$ . We take  $a_n = \frac{(-1)^n}{n!}$ ;  $c_n = \frac{1}{2^{n-1}}$ . Then  $\sum_{n=3}^\infty \frac{1}{2^n}$  converges. Therefore by above theorem, the series  $\sum a_n$  converges absolutely.

**Theorem 3.3.3** (Limit comparison test). Let  $\{a_n\}$  and  $\{b_n\}$  be two sequences of positive numbers. Then

- a) if  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = c > 0$ ,  $\sum a_n$  and  $\sum b_n$  both converge or diverge together;
- b) if  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 0$  and  $\sum b_n$  converges, then  $\sum a_n$  converges.
- c) if  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \infty$  and  $\sum b_n$  diverges, then  $\sum a_n$  diverges.

*Proof.*

(a) As  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = c > 0$ , for  $\epsilon = \frac{c}{2} > 0$ , there exists  $N \in \mathbb{N}$  such that

$$n \geq N \implies \left| \frac{a_n}{b_n} - c \right| < \frac{c}{2}.$$

Thus, for  $n \geq N$ ,

$$\frac{-c}{2} \leq \frac{a_n}{b_n} - c \leq \frac{c}{2}$$

or equivalently

$$\frac{cb_n}{2} \leq a_n \leq \frac{3cb_n}{2}.$$

Hence the conclusion follows from the comparison test.

(b) Given that  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 0$ . Hence for  $\epsilon = \frac{1}{2}$ , there exists  $N \in \mathbb{N}$  such that

$$n \geq N \implies \frac{a_n}{b_n} < \frac{1}{2}$$

or equivalently,

$$n \geq N \implies a_n \leq \frac{b_n}{2}.$$

Thus the desired conclusion follows from the comparison test.

(c) Here the proof follows by changing the roles of  $a_n$  and  $b_n$  in (b) and applying comparison test.

**Problem 25.** *Test the convergence of*

$$(1) \sum_{n=1}^{\infty} \frac{2n+1}{(n+1)^2} \quad (2) \sum_{n=1}^{\infty} \frac{1}{2^n - 1} \quad (3) \sum \frac{e^{-n}}{n^2} \quad (4) \sum \frac{e^{-n}}{n}$$

**Solution:**

- 1) Here  $a_n = \frac{2n+1}{(n+1)^2}$ . Let  $b_n = \frac{1}{n}$ . Then  $\frac{a_n}{b_n} = \frac{\left(\frac{2n+1}{(n+1)^2}\right)}{\frac{1}{n}} = \frac{2n^2 + n}{n^2 + 2n + 1} \rightarrow 2$  as  $n \rightarrow \infty$ . Further,  $\sum \frac{1}{n}$  diverges. Thus by limit comparison theorem, the given series diverges.
- 2) Here  $a_n = \frac{1}{2^n - 1}$ . Let  $b_n = \frac{1}{2^n}$ . Then  $\frac{a_n}{b_n} = \frac{2^n}{2^n - 1} \rightarrow 1$ . Further,  $\sum \frac{1}{2^n}$  converges and hence the given series converges.
- 3) Here  $a_n = \frac{e^{-n}}{n^2}$  and  $b_n = \frac{1}{n^2}$ . Then  $\frac{a_n}{b_n} = e^{-n} \rightarrow 0$  as  $n \rightarrow \infty$ . Further,  $\sum \frac{1}{n^2}$  converges and hence the given series converges.
- 4) Here  $a_n = \frac{e^{-n}}{n}$  and  $b_n = \frac{1}{n^2}$ . Then  $\frac{a_n}{b_n} = ne^{-n} \rightarrow 0$  as  $n \rightarrow \infty$ . Further,  $\sum \frac{1}{n^2}$  converges and hence the given series converges.

**Theorem 3.3.4. Ratio test:** Let  $\sum_{n=1}^{\infty} a_n$  be a series of real numbers. Let

$$a = \liminf_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| \quad \text{and} \quad A = \limsup_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|.$$

Then

a)  $\sum_1^{\infty} a_n$  converges absolutely if  $A < 1$ ;

b)  $\sum_1^{\infty} |a_n|$  diverges if  $a > 1$ ;

c) the test fails in all other cases.

*Proof.* a) If  $A < 1$ , choose  $B$  such that  $A < B < 1$ . Then there exists an  $\epsilon > 0$  such that  $B = A + \epsilon$  and also  $N \in \mathbb{N}$  such that  $\left| \frac{a_{n+1}}{a_n} \right| \leq B$  for all  $n \geq N$ . Further, for any  $k \in \mathbb{N}$ ,

$$\left| \frac{a_{N+k}}{a_N} \right| = \prod_{i=1}^k \left| \frac{a_{N+i}}{a_{N+i-1}} \right| \leq \prod_{i=1}^k B = B^k.$$

Thus  $|a_{N+k}| \leq B^k |a_N|$ ,  $k \in \mathbb{N}$ . But  $\sum_{k=0}^{\infty} |a_N| B^k < \infty$  as  $B < 1$ . Thus by comparison test, the series  $\sum_1^{\infty} a_n$  converges.

b) If  $a > 1$ , choose  $b$  such that  $1 < b < a$ . There exists  $N \in \mathbb{N}$  such that  $\left| \frac{a_{n+1}}{a_n} \right| \geq b$  for all  $n \geq N$ . Further, for any  $k \in \mathbb{N}$ ,

$$\left| \frac{a_{N+k}}{a_N} \right| = \prod_{i=1}^k \left| \frac{a_{N+i}}{a_{N+i-1}} \right| \geq \prod_{i=1}^k b = b^k.$$

Thus  $|a_{N+k}| \geq |a_N|$ ,  $k \in \mathbb{N}$ . But, as  $b > 1$ ,  $\sum_{k=0}^{\infty} a_N b^k$  diverges. Thus, again, by the comparison test, the series  $\sum_1^{\infty} a_n$  diverges.

c) Case1:  $a = A = 1$  Consider the series  $\sum \frac{1}{n}$ . Here  $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = 1$ . But  $\sum \frac{1}{n}$  diverges. For the series  $\sum \frac{1}{n^2}$ , which converges, again  $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = 1$ .

Case 2:  $A > 1$  If we consider the series  $\sum 2^n$  then  $A = 2 > 1$  and the series diverges. If we take

$$s = 1 + 2 + \frac{1}{5} + \frac{2}{5} + \left(\frac{1}{5}\right)^2 + 2\left(\frac{1}{5}\right)^2 + \left(\frac{1}{5}\right)^3 + \dots$$

Then it is easy to see that the series converges as

$$s = 1 + \left(\frac{1}{5}\right) + \left(\frac{1}{5}\right)^3 + \dots + 2 + 2\left(\frac{1}{5}\right) + 2\left(\frac{1}{5}\right)^2 + 2\left(\frac{1}{5}\right)^3 + \dots$$

But  $\frac{a_{n+1}}{a_n} = \left\{ 2, \frac{1}{10}, 2, \frac{1}{10}, \dots \right\}$  and  $A = \limsup \left| \frac{a_{n+1}}{a_n} \right| = 2$  and  $a = \liminf \left| \frac{a_{n+1}}{a_n} \right| = 1/10$ .

**Problem 26.** Test the convergence of

$$(1) \sum_1^{\infty} \frac{n^n}{n!} \quad (2) \sum_0^{\infty} \frac{a^n}{n!}, a \in \mathbb{R}$$

**Solution:**

a) Here

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^{n+1}}{(n+1)!} \frac{n!}{n^n} = \left( \frac{n+1}{n} \right)^n = \left( 1 + \frac{1}{n} \right)^n \rightarrow e,$$

which is greater than 1. So  $a = A = e > 1$ . Thus the given series diverges.

b) Here

$$\frac{a_{n+1}}{a_n} = \frac{a^{n+1}}{(n+1)!} \frac{n!}{a^n} = \frac{a}{n+1} \rightarrow 0.$$

Therefore  $a = A = 0 < 1$ . Thus, for all  $x \in \mathbb{R}$ , the given series converges.

**Theorem 3.3.5. Root test:** Let  $\sum_{n=1}^{\infty} a_n$  be a series of real numbers and

$$A = \limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|}.$$

Then

a) the series converges absolutely if  $A < 1$ ;

b) the series diverges if  $A > 1$ ;

c) the test fails if  $A = 1$ .

*Proof.* a) If  $A < 1$ , choose  $B$  such that  $A < B < 1$ . Then there exists  $N \in \mathbb{N}$  such that  $\sqrt[n]{|a_n|} < B$  for all  $n \geq N$ . This implies  $|a_n| < B^n$  for all  $n \geq N$ . As  $B < 1$ , the series converges by comparison test.

b) If  $A > 1$ , there exists infinitely many  $n \in \mathbb{N}$  such that  $\sqrt[n]{|a_n|} > 1$ . But this implies that  $|a_n| > 1$  for infinitely many values of  $n$  and hence  $a_n \not\rightarrow 0$ , i.e.,  $\sum a_n$  diverges.

c) Consider the series  $\sum \frac{1}{n}$ . Here  $A = 1$  and the series diverges. On the other hand, for the series  $\sum \frac{1}{n^2}$ , again  $A = 1$ , but the series converges.

**Problem 27.** Test the convergence of

$$(1) \sum_1^{\infty} \frac{a^n}{n}, a \in \mathbb{R} \quad (2) \sum a_n, a_n = \begin{cases} \frac{n}{4^n} & n \text{ is odd} \\ \frac{1}{2^n} & n \text{ is even} \end{cases} \quad (3) \sum 3^{-n-(-1)^n}$$

**Solution:**

- 1) Here  $a_n = \frac{a^n}{n}$ . Therefore,  $\sqrt[n]{\left|\frac{a^n}{n}\right|} = \left|\frac{a}{\sqrt[n]{n}}\right| \rightarrow |a|$ . Thus the series converges for  $|a| < 1$  and diverges for  $|a| > 1$ .
- 2) Here  $\limsup_{n \rightarrow \infty} \sqrt[n]{a_n} = \frac{1}{2}$ . Therefore the series converges.
- 3) It is not difficult to see that  $\limsup |a_n|^{\frac{1}{n}} = 1/3$ . However ratio test fails in this case.

**Problem 28.** Test the convergence of  $\sum a_n$  where

$$a_n = \begin{cases} 2^{-n} & n \text{ odd} \\ 2^{-n+2} & n \text{ even} \end{cases}$$

It is easy to see that

$$\left| \frac{a_{n+1}}{a_n} \right| = \begin{cases} \frac{2^{-(n+1)+2}}{2^{-n}} & n \text{ odd} \\ \frac{2^{-(n+1)}}{2^{-n+2}} & n \text{ even} \end{cases} = \begin{cases} 2 & n \text{ odd} \\ \frac{1}{8} & n \text{ even} \end{cases}$$

$$\therefore \limsup \frac{|a_{n+1}|}{|a_n|} = 2.$$

So the Ratio test fails. However we can see

$$\limsup |a_n|^{1/n} = \lim |a_n|^{1/n} = \frac{1}{2}.$$

So the root test implies that the series converges but ratio test is inconclusive.

### 3.4 Alternating series

In this section we will study the convergence properties of series with each term having opposite sign of previous term. Such series are called alternating series.

**Definition 3.4.1.** An alternating series is an infinite series whose terms alternate in sign.

We have the following Leibniz test for the convergence of alternating series.

**Theorem 3.4.2.** Suppose  $\{a_n\}$  is a sequence of positive numbers such that

- (a)  $a_n \geq a_{n+1}$  for all  $n \in \mathbb{N}$  and
- (b)  $\lim_{n \rightarrow \infty} a_n = 0$ ,

then the alternating series  $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$  converges.

*Proof.* Consider the partial sums with odd index,  $s_1, s_3, s_5, \dots$ . Now, for any  $n \in \mathbb{N}$ ,

$$s_{2n+1} = s_{2n-1} - a_{2n} + a_{2n+1} \leq s_{2n-1} \text{ (by (a)).}$$

Thus the sequence  $\{s_{2n-1}\}_1^\infty$  forms a non-increasing sequence. Also, notice that

$$s_{2n-1} = \sum_{i=1}^{n-1} (a_{2i-1} - a_{2i}) + a_{2n-1}.$$

Since each quantity in the parenthesis is non-negative and  $a_{2n-1} > 0$ , the sequence  $\{s_{2n-1}\}$  is bounded below by 0. Hence  $\{s_{2n-1}\}_1^\infty$  is convergent.

Now, consider the partial sums with even index,  $s_2, s_4, s_6, \dots$ . For any  $n \in \mathbb{N}$ ,

$$s_{2n+2} = s_{2n} + a_{2n+1} - a_{2n+2} \geq s_{2n} \text{ (by (a))}.$$

Thus the sequence  $\{s_{2n}\}_1^\infty$  forms a non-decreasing sequence. Further,

$$s_{2n} = a_1 - \sum_{i=1}^{n-1} (a_{2i} - a_{2i+1}) - a_{2n} \leq a_1,$$

which means that  $s_{2n}$  is bounded above by  $a_1$ . Therefore,  $\{s_{2n}\}$  is convergent.

Let  $L = \lim s_{2n}$  and  $M = \lim s_{2n-1}$ . By ((b)),

$$0 = \lim a_{2n} = \lim (s_{2n} - s_{2n-1}) = L - M.$$

Thus  $L = M$  and hence the alternating series  $\sum_{n=1}^\infty (-1)^{n+1} a_n$  converges.

**Problem 29.** *Test the convergence of*

$$(1) \sum_{n=1}^\infty \frac{(-1)^{n+1}}{n} \quad (2) \sum_{n=1}^\infty (1 - 2^{1/n})(-1)^{n+1} \quad (3) \sum_{n=1}^\infty (-1)^{n+1} 2^{1/n}$$

1. The  $a'_n$ s of this series satisfies the hypothesis of the above theorem and hence the series converges.
2. Here  $a_n = 1 - 2^{1/n} \rightarrow 0$  as  $n \rightarrow \infty$ . Also  $a_n \geq a_{n+1}$  for all  $n \in \mathbb{N}$ . Hence the series converges.
3. Here  $a_n = 2^{1/n} \rightarrow 1$  as  $n \rightarrow \infty$ . Hence the above theorem does not apply. Anyhow, one can show that the series diverges.

**Definition 3.4.3.** Let  $\sum_{n=1}^\infty a_n$  be a series of real numbers. If  $\sum_{n=1}^\infty a_n$  converges but  $\sum_{n=1}^\infty |a_n|$  diverges, we say that  $\sum_{n=1}^\infty a_n$  converges conditionally.

**Example 3.4.4.** The series  $\sum_{n=1}^\infty \frac{(-1)^{n+1}}{n}$  converges conditionally.