MTL108: Problem Set-12

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IPSES refers to INTRODUCTION TO PROBABILITY AND STATISTICS FOR ENGINEERS AND SCIENTISTS book, by Ross.

Instructions: For each problem, clearly state the test statistic, the exact distribution of the test statistic under the null hypothesis, and the rejection region. Where applicable, justify the use of the Generalized Likelihood Ratio Test (GLRT) for composite hypotheses.

Problem 1: Testing the Mean of a Normal Population

Let $X_1, X_2, \ldots, X_n$ be a random sample from a Normal distribution $N(\mu, \sigma^2)$.

(a) **Known Variance:** Assume the population variance $\sigma^2 = \sigma_0^2$ is known. Use the Neyman-Pearson Lemma to derive the Most Powerful (MP) test of size α for the simple hypotheses $H_0 : \mu = \mu_0$ against $H_1 : \mu = \mu_1$, where $\mu_1 > \mu_0$. Conclude by explaining why this test is Most Powerful (MP) for the one-sided composite alternative $H_1 : \mu > \mu_0$.

(b) **Unknown Variance:** Assume the population variance σ^2 is unknown. Because both hypotheses are now composite, the strict Neyman-Pearson Lemma does not apply. Instead, derive the Likelihood Ratio Test (LRT) for $H_0 : \mu = \mu_0$ against the two-sided alternative $H_1 : \mu \neq \mu_0$.

Problem 2: Testing the Variance of a Normal Population

Let $X_1, X_2, \ldots, X_n$ be a random sample from a Normal distribution $N(\mu, \sigma^2)$.

(a) **Known Mean:** Assume the population mean $\mu = \mu_0$ is known. Use the Neyman-Pearson Lemma to derive the MP test of size α for $H_0 : \sigma^2 = \sigma_0^2$ against $H_1 : \sigma^2 = \sigma_1^2$ (assume $\sigma_1^2 > \sigma_0^2$). Express your critical region in terms of a strictly chi-squared (χ^2) distributed test statistic.

(b) **Unknown Mean:** Assume the population mean μ is unknown. Derive the LRT for $H_0 : \sigma^2 = \sigma_0^2$ against the two-sided alternative $H_1 : \sigma^2 \neq \sigma_0^2$. Define the test statistic and its exact degrees of freedom under the null hypothesis.

Problem 3: Testing the Parameter of a Poisson Distribution

Let $X_1, X_2, \ldots, X_n$ be a random sample from a Poisson distribution with unknown rate parameter λ.

(a) **One-Sided Test:** Use the Neyman-Pearson Lemma to find the MP test of size α for testing $H_0 : \lambda \leq \lambda_0$ against $H_1 : \lambda > \lambda_0$.

(b) **Two-Sided Test:** Propose an exact test for $H_0 : \lambda = \lambda_0$ against $H_1 : \lambda \neq \lambda_0$. Define the test statistic and explain how to determine the critical values c_1 and c_2 using the Poisson CDF to maintain a significance level of approximately α.

Problem 4: Difference Between Two Normal Means

Let $X_1, \ldots, X_{n_1}$ be a random sample from $N(\mu_1, \sigma_1^2)$ and $Y_1, \ldots, Y_{n_2}$ be an independent random sample from $N(\mu_2, \sigma_2^2)$. Assume that the population variances are equal ($\sigma_1^2 = \sigma_2^2 = \sigma^2$).

(a) **Equal and Known Variance:** Assume the common variance σ^2 is a known constant. Construct a size α test for $H_0 : \mu_1 - \mu_2 = 0$ against the one-sided alternative $H_1 : \mu_1 - \mu_2 > 0$. Provide the test statistic and the critical value.

(b) **Equal but Unknown Variance:** Assume the common variance σ^2 is unknown. Derive the LRT for testing $H_0 : \mu_1 - \mu_2 = 0$ against $H_1 : \mu_1 - \mu_2 \neq 0$.