MTL108
Classical (Naive) Probability

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Definition 1 (Random Experiment)**.** A random experiment is a process or action that produces an outcome, where the result cannot be predicted with certainty in advance.

E.g., tossing a coin, rolling a die, or drawing a card from a deck.

Definition 2 (Sample Space)**.** The sample space, denoted by Ω, is the set of all possible outcomes of a random experiment.

E.g., for a coin toss $\Omega = \{\text{Head}, \text{Tail}\}$.

Definition 3 (Event)**.** An event is a subset of the sample space. If Ω is the sample space, then an event E is such that $E \subseteq \Omega$.

E.g., when rolling a die, $E = \{2, 4, 6\}$ represents the event of obtaining an even number.

Definition 4 (Naïve Probability)**.** Consider a random experiment. If **(a)** the sample space is a finite set; and **(b)** all the outcomes are equally likely. Then, the probability of an event E is defined as

$$P(E) = \frac{|E|}{|\Omega|},$$

where $|E|$ is the number of outcomes favorable to E and $|\Omega|$ is the total number of possible outcomes in the sample space.

Problem 1 What is the probability of rolling an even number with a single fair six-sided die?

- **Solution**: The sample space for a single die consists of the outcomes $\{1, 2, 3, 4, 5, 6\}$, so the total number of outcomes is 6. The event of rolling an even number corresponds to the outcomes $\{2, 4, 6\}$, which has 3 favorable outcomes. Thus, the probability is:

$$P(\text{even}) = \frac{\text{Number of even outcomes}}{\text{Total outcomes}} = \frac{3}{6} = \frac{1}{2}.$$

Problem 2 What is the probability of rolling a number greater than 4 with a single fair six-sided die?

- **Solution**: The sample space is $\{1, 2, 3, 4, 5, 6\}$, with 6 total outcomes. The event of rolling a number greater than 4 corresponds to $\{5, 6\}$, which has 2 favorable outcomes. Thus, the probability is:

$$P(\text{greater than 4}) = \frac{\text{Number of outcomes} > 4}{\text{Total outcomes}} = \frac{2}{6} = \frac{1}{3}.$$

Sample Space for Two Dice

When rolling two fair six-sided dice, each die has faces numbered 1 through 6. The sample space S consists of all possible ordered pairs (a, b), where a is the number on the first die and b is the number on the second die. Since each die has 6 possible outcomes, the total number of outcomes in the sample space is:

$$6 \times 6 = 36.$$

The sample space can be represented as:

$$S = \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,1), (2,2), \ldots, (6,5), (6,6)\}.$$

Each outcome is equally likely, with probability $\frac{1}{36}$.

Problem 3 What is the probability that the sum of the numbers on two fair six-sided dice is 7?

- **Solution**: The sample space has $6 \times 6 = 36$ possible outcomes. We need to find all pairs (a, b) where $a + b = 7$. The possible pairs are:

  $$(1,6), (2,5), (3,4), (4,3), (5,2), (6,1).$$

  There are 6 favorable outcomes. Thus, the probability is:

  $$P(\text{sum} = 7) = \frac{\text{Number of pairs with sum 7}}{\text{Total outcomes}} = \frac{6}{36} = \frac{1}{6}.$$

Problem 4 What is the probability that the sum of the numbers on two fair six-sided dice is at least 10?

- **Solution**: The sample space has 36 outcomes. The event of a sum at least 10 means the sum is 10, 11, or 12. We list the favorable pairs:
  - Sum = 10: $(4,6), (5,5), (6,4)$ (3 pairs).
  - Sum = 11: $(5,6), (6,5)$ (2 pairs).
  - Sum = 12: $(6,6)$ (1 pair).

  Total favorable outcomes: $3 + 2 + 1 = 6$. Thus, the probability is:

  $$P(\text{sum} \geq 10) = \frac{\text{Number of pairs with sum } \geq 10}{\text{Total outcomes}} = \frac{6}{36} = \frac{1}{6}.$$

Problem 5 What is the probability that both dice show the same number when rolling two fair six-sided dice?

- **Solution**: The sample space has 36 outcomes. The event that both dice show the same number corresponds to the pairs $(1,1), (2,2), (3,3), (4,4), (5,5), (6,6)$, which gives 6 favorable outcomes. Thus, the probability is:

  $$P(\text{same number}) = \frac{\text{Number of pairs with same number}}{\text{Total outcomes}} = \frac{6}{36} = \frac{1}{6}.$$

Problem 6 What is the probability that the first die shows an odd number and the second die shows an even number?

- **Solution**: The sample space has 36 outcomes. The first die shows an odd number: $\{1, 3, 5\}$ (3 outcomes). The second die shows an even number: $\{2, 4, 6\}$ (3 outcomes). Since the dice are independent, the number of favorable pairs is:

$$3 \times 3 = 9.$$

The pairs are:

$$(1, 2), (1, 4), (1, 6), (3, 2), (3, 4), (3, 6), (5, 2), (5, 4), (5, 6).$$

Thus, the probability is:

$$P(\text{first odd, second even}) = \frac{\text{Number of favorable pairs}}{\text{Total outcomes}} = \frac{9}{36} = \frac{1}{4}.$$

Problem 7 What is the probability that the sum of the numbers on two fair six-sided dice is a prime number?

- **Solution**: The probability of an event E is given by:

$$P(E) = \frac{\text{Number of favorable outcomes}}{\text{Total number of outcomes}}.$$

The total number of outcomes is 36 (from the sample space). We need to find the number of pairs (a, b) where the sum $a + b$ is a prime number. A prime number is a positive integer greater than 1 that is divisible only by 1 and itself. The possible sums when rolling two dice range from 2 (i.e., $1 + 1$) to 12 (i.e., $6 + 6$). The prime numbers in this range are:

$$\{2, 3, 5, 7, 11\}.$$

We list the pairs (a, b) for each prime sum:

  - Sum = 2: $(1, 1)$ (1 pair).
  - Sum = 3: $(1, 2), (2, 1)$ (2 pairs).
  - Sum = 5: $(1, 4), (2, 3), (3, 2), (4, 1)$ (4 pairs).
  - Sum = 7: $(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)$ (6 pairs).
  - Sum = 11: $(5, 6), (6, 5)$ (2 pairs).

The sum of 13 is not possible since the maximum sum is 12. Thus, the favorable outcomes are:

$$(1, 1), (1, 2), (2, 1), (1, 4), (2, 3), (3, 2), (4, 1), (1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1), (5, 6), (6, 5).$$

Counting these, we have:

$$1 + 2 + 4 + 6 + 2 = 15 \text{ favorable outcomes.}$$

Therefore, the probability that the sum is prime is:

$$P(\text{sum is prime}) = \frac{\text{Number of pairs with prime sum}}{\text{Total outcomes}} = \frac{15}{36} = \frac{5}{12}.$$

Extending Naïve Probability to Geometric Settings

In classical (naïve) probability, the probability of an event is defined as

$$P(E) = \frac{\text{Number of favorable outcomes}}{\text{Total number of outcomes}}.$$

When moving to geometric settings, the sample space and events are no longer finite sets but subsets of a geometric region (such as intervals, areas, or volumes).

Definition 5. The principle of symmetry leads to the definition:

$$P(E) = \frac{\mu(E)}{\mu(\Omega)},$$

where μ is a suitable measure (length, area, or volume), Ω is the total geometric region under consideration, and $E \subseteq \Omega$ is the event of interest.

Example 1. Uniform point on a line segment. Suppose a point is chosen uniformly at random from the interval $[0,1]$. Find the probability that it lies in $[0.25, 0.5]$.

$$P\big([0.25, 0.5]\big) = \frac{\mu([0.25, 0.5])}{\mu([0,1])} = \frac{0.5 - 0.25}{1 - 0} = 0.25.$$

Example 2. Random point in a square. A point is chosen uniformly at random inside the unit square $\Omega = [0,1] \times [0,1]$. Find the probability that the point lies inside the circle of radius $1/2$ centered at $(0.5, 0.5)$.

$$P(E) = \frac{\mu(E)}{\mu(\Omega)} = \frac{\pi(1/2)^2}{1} = \frac{\pi}{4} \approx 0.785.$$

Bertrand's Paradox (Random chord in a circle)

A chord is drawn at random in a unit circle. What is the probability that its length exceeds $\sqrt{3}$?

For a unit circle (radius $R = 1$), a chord has length

$$L = 2R \sin\left(\frac{\theta}{2}\right) = 2 \sin\left(\frac{\theta}{2}\right),$$

where $\theta \in [0, \pi]$ is the central angle subtended by the chord. We call a chord *long* if $L > \sqrt{3}$. This is equivalent to

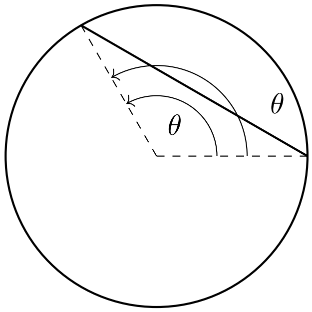
$$2\sin\left(\tfrac{\theta}{2}\right) > \sqrt{3} \iff \sin\left(\tfrac{\theta}{2}\right) > \frac{\sqrt{3}}{2} \iff \frac{\theta}{2} > \frac{\pi}{3} \iff \theta > \frac{2\pi}{3}.$$

Interpretation A: Random Endpoints on the Circumference

Construction. Pick two points independently and uniformly on the circumference; equivalently, fix one point and choose a random angle $\theta \sim \text{Uniform}[0, \pi]$ to determine the other endpoint.

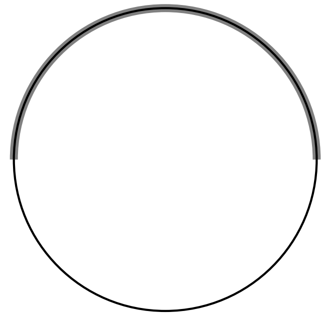
Probability.

$$\mathbb{P}(L > \sqrt{3}) = \mathbb{P}\left(\theta > \frac{2\pi}{3}\right) = 1 - \frac{2\pi/3}{\pi} = \frac{1}{3}.$$



Endpoints: θ uniform in $[0, \pi]$.
Long chord iff $\theta > \frac{2\pi}{3}$.

(a) Random endpoints.



Heuristic idea: angles above 120°
yield $L > \sqrt{3}$.

(b) Angles > 120° give long chords.

Interpretation B: Random Midpoint Uniform in the Disk

Construction. Choose the chord by choosing its midpoint uniformly in the area of the disk and orienting the chord perpendicular to the radius through that midpoint. If the midpoint is at distance r from the center, then

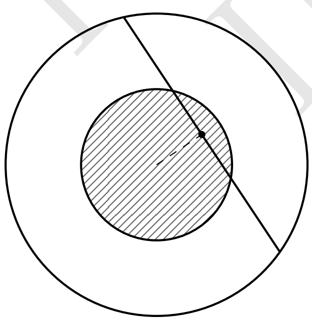
$$L = 2\sqrt{1 - r^2}.$$

Long iff

$$2\sqrt{1 - r^2} > \sqrt{3} \iff 1 - r^2 > \frac{3}{4} \iff r < \frac{1}{2}.$$

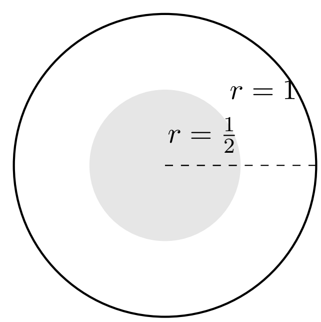
Since midpoints are uniform *in area*, the probability is area ratio:

$$\mathbb{P}(L > \sqrt{3}) = \frac{\pi(1/2)^2}{\pi \cdot 1^2} = \frac{1}{4}.$$



Shaded region: $r < \frac{1}{2}$ gives $L > \sqrt{3}$.

(a) Uniform midpoint in area ⇒ area test.



$$\mathbb{P} = \frac{\text{area}(r < 1/2)}{\text{area}(r < 1)} = \frac{1}{4}.$$

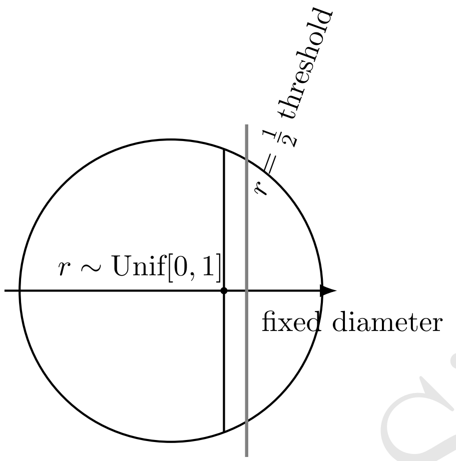
(b) Area ratio: 1/4.

Interpretation C: Random Chord Perpendicular to a Fixed Radius with Uniform Point Along the Radius

Construction. Fix a radius (or diameter). Choose a point uniformly at random along this radius (distance $r \sim \text{Uniform}[0, 1]$ from the center). Draw the chord through this point perpendicular to the radius.

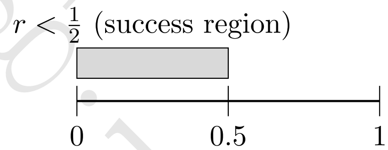
As before, the chord length is $L = 2\sqrt{1-r^2}$. The *long* condition is $r < \frac{1}{2}$. But now the distribution of r is *uniform in length* on $[0, 1]$, not in area; hence

$$\mathbb{P}(L > \sqrt{3}) = \mathbb{P}\big(r < \tfrac{1}{2}\big) = \frac{1}{2}.$$



Length test identical; distribution of r differs.

(a) Uniform along a radius: linear measure.



$$\mathbb{P} = \frac{1/2}{1} = 1/2.$$

(b) Linear ratio: 1/2.

Takeaway (Why the Paradox?)

Each method is symmetric but uses a different underlying notion of "uniform":

- Uniform *angle* (endpoints on circumference) $\Rightarrow \mathbb{P} = 1/3$.
- Uniform *area* of midpoints $\Rightarrow \mathbb{P} = 1/4$.
- Uniform *length* along a fixed radius $\Rightarrow \mathbb{P} = 1/2$.

Without a precise sampling scheme, "random chord" is ambiguous, yielding different answers.

References

[1] Blitzstein, J. K., & Hwang, J. (2019). *Introduction to probability.* Chapman and Hall/CRC.

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