Feb 13

Markov Inequality

Theorem: let X be a non-negative RV with finite mean. $\mu = E(X)$.

Then for any fixed $a > 0$,

$$P(X \geq a) \leq \frac{E(X)}{a}$$

Proof: let X is a cont RV with PDF $f(\cdot)$.

Now $E(X) = \int_0^\infty x f(x)\,dx$

$$= \int_0^a x f(x)\,dx + \int_a^\infty x f(x)\,dx$$

Observe that

(ii) $$\int_0^a x f(x)\, dx \geq 0$$

(iii) $$\underline{\int_a^\infty x\, f(x)\, dx} \geq \int_a^\infty a\, f(x)\, dx$$

$$= a \cdot \underline{\int_a^\infty f(x)\, dx}$$

(III) $$\int_a^\infty x\, f(x)\, dx \geq a\, \mathbb{P}(X \geq a)$$

From (I) (II) & (III)

$$\mathbb{E}(X) \geq \int_a^\infty x\, f(x)\, dx$$

$$\geq \underline{a\, \mathbb{P}(X \geq a)}$$

$$\Rightarrow \quad \mathbb{E}(X) \qquad \mathbb{P}(X \geq a)$$

$$\Rightarrow \frac{E(X)}{a} \geq P(X \geq a)$$

$$\Rightarrow \boxed{P(X \geq a) \leq \frac{E(X)}{a}}$$

Discrete ?

(i) Replace PDF with PMF

(ii) Replace $\int$ with Σ

Chebychev's inequality

Theorem: let X be a RV with mean μ and variance σ^2. Then for any $\varepsilon > 0$

$$P(|X - \mu| \geq \varepsilon) \leq \frac{\sigma^2}{\ }$$

$$\boxed{\mathbb{P}(|X-\mu| \geq \varepsilon) \leq \frac{\sigma^2}{\varepsilon^2}}$$

Proof / Hint Cont. with PDF f

$$\mathbb{P}(|X-\mu| \geq \varepsilon)$$

$$= \mathbb{P}\left((X-\mu)^2 \geq \varepsilon^2\right)$$

$$\mathbb{E}\left[(X-\mu)^2\right] = \sigma^2$$

$$= \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx$$

$$= \int_{(x-\mu)^2 \geq \varepsilon^2} (x-\mu)^2 f(x)\, dx$$

$$+ \int_{(x-\mu)^2 \leq \varepsilon^2} (x-\mu)^2 f(x)\, dx$$

Lemma:

$$\mathbb{P}(|X-\mu| \leq \epsilon) \geq 1 - \frac{\sigma^2}{\varepsilon^2}$$

Eg. $\mu = 500$. $\sigma = 50$

$\underbrace{\{X > 750\}}_{A}$

$750 - 500 = 250$

$\underbrace{\{|X-\mu| \geq 250\}}_{B}$ ✓

$\{|X - 500| \geq 250\}$

$= \{ |X$

$= \{ X \leq 250$ or $X \geq 750 \}$

$= \{ X \leq 250 \} \cup \{ X \geq 750 \}$

Then $A \subset B$

$\Rightarrow \mathbb{P}(A) \leq \mathbb{P}(B)$

Convergence of RVs

(1) Convergence in Probability:

Let $\{X_n\}_{n \geq 1}$ be a sequence of RVs. The seq $\{X_n\}_{n \geq 1}$ is said to converge in is said to converge in probability to a RV X if for any $\varepsilon > 0$

$$\lim_{n \to \infty} \mathbb{P}(|X_n - X| > \varepsilon) = 0.$$

It is denoted by

$$X_n \xrightarrow[n \to \infty]{P} X.$$

Eg: Let $X_1, \dots, X_n$ be IID RVs from Uniform$(0, \theta)$

$$Y_n = \max\{X_1, \dots, X_n\}.$$

Then $\{Y_n\}_{n \geq 1}$ is a seq of RVs. The limiting ... of Y_n is

CDF of Y_n is

$$\lim_{n\to\infty} \mathbb{P}(Y_n \le y) = \begin{cases} 0 & \text{if } y < \theta \\ 1 & \text{if } y \geq \theta \end{cases}$$

Let Y be a RV degenerate at θ. Then

$$\lim_{n\to\infty} \mathbb{P}(|Y_n - Y| > \varepsilon) = 0$$

or

$$\lim_{n\to\infty} \mathbb{P}(|Y_n - \theta| > \varepsilon) = 0$$

Eg: Let $X_1, \dots X_n$ be IID RVs with mean μ and variance σ^2.

Define $\bar{X}_n = \dfrac{X_1 + \cdots + X_n}{n}$

Then $\mathbb{E}(\bar{X}_n) = \mu$

$$\mathrm{Var}(\bar{X}_n) = \sigma^2/n$$

Use Chebychev's inequality

$$\mathbb{P}(|\bar{X}_n - \mu| > \varepsilon) \leq \frac{\mathrm{Var}(\bar{X}_n)}{\varepsilon^2}$$

$$\Rightarrow \mathbb{P}(|\bar{X}_n - \mu| > \varepsilon) \leq \frac{\sigma^2}{n\varepsilon^2}$$

$$\Rightarrow \lim_{n\to\infty} \mathbb{P}(|\bar{X}_n - \mu| > \varepsilon) = 0$$

$$\Rightarrow \quad \bar{X}_n \xrightarrow[n\to\infty]{p} \mu$$