Jan 30

Continuous random variables

Easiest way to characterise using PDF.

Recall: a function $f: \mathbb{R} \to \mathbb{R}$ is a valid pdf if

(i) $f(x) \geq 0$

(ii) $\int_{-\infty}^{\infty} f(x)\, dx = 1$

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(1) Uniform random variable:

$X \sim \text{Uniform}(a, b)$

if its PDF is given

by

$$f(x) = \begin{cases} \frac{1}{b-a} & ; \text{if } x \in (a, b) \\ 0 & \text{otherwise} \end{cases}$$

or.

$$f(x) = \begin{cases} \frac{1}{b-a} & ; \text{if } x \in [a, b] \\ 0 & , \text{otherwise} \end{cases}$$

(ii) Exponential:

let $\lambda > 0$.

$$\boxed{\int_0^\infty \underbrace{\lambda e^{-\lambda x}} dx = 1}$$

Def: X is said to follow exponential distribution with Parameter λ if it's PDF is given by

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

Notation:

$$X \sim \text{Exponential}(\lambda)$$

Mean:

$$E(X) = \int_{-\infty}^{\infty} x f(x)\, dx$$

$$= \int_{0}^{\infty} x \cdot \lambda e^{-\lambda x}\, dx$$

$$= \frac{1}{\lambda}$$

$$Var(X) =$$

Normal distribution

$$\int_{-\infty}^{\infty} e^{-x^2} dx$$

gamma integrals.

Fact: $$\boxed{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} = 1}$$

Def: A random variable X is said to follow normal (Gaussian) distribution with parameters μ and σ^2 if its PDF is

given by $\quad (x-\mu)^2$

given by

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma}\, e^{-\frac{\quad}{2\sigma^2}}$$

where $x \in \mathbb{R}$

$$E(X) = \int_{-\infty}^{\infty} x\, f(x) = \mu$$

$$Var(X) = \sigma^2$$

Remark: The parameters μ & σ^2 are mean & variance of normal distribution.

Notation:

$$X \sim N(\mu, \sigma^2)$$

mean parameter variance parameter.

<u>Lemma</u>: Let $X \sim N(\mu, \sigma^2)$

and $Z = \frac{X-\mu}{\sigma}$.

Then

$Z \sim N(0, 1)$.

<u>Pf</u>: Using CDF,

$\mathbb{P}(Z \leq t)$

$= \mathbb{P}\left(\frac{X-\mu}{\sigma} \leq t\right)$

$= \mathbb{P}(X \leq \mu + \sigma t)$

$\int^{\mu+\sigma t} \quad e^{-\frac{(x-\mu)^2}{2\sigma^2}} \, dx$

$$= \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}\,\sigma} e \ldots$$

transformation

$$x \to \frac{x-\mu}{\sigma} = y$$

$$= \underbrace{\int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} \, dy}_{\text{CDF of } N(0,1)}$$

<u>Remark:</u> Let X be a RV and $g : \mathbb{R} \to \mathbb{R}$ be a $(\mathbb{R}, \mathcal{B}(\mathbb{R})) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ measurable function.

measurable function,
then $g(X)$ is also a RV.

Def: (Existence of expectation)

let X be a RV and g be a measurable function. The expectation of $g(X)$ is said to exist if $\mathbb{E}[|g(X)|] < \infty$.

Ex: let X be a RV with support $\mathcal{X} = \{1, 2, \cdots\}$ and

PMF

$$p(x) = \begin{cases} \dfrac{1}{c x^2} & \text{if } x \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$

where $$c = \sum_{i=1}^{\infty} \frac{1}{x^2} < \infty$$

$$\boxed{= \pi^2/6 \,?}$$

So, $p(x)$ is a valid PMF.

$$\mathbb{E}(X) = \sum_{x=1}^{\infty} x \cdot \frac{1}{c x^2}$$

$$= \frac{1}{c} \boxed{\sum_{x=1}^{\infty} \frac{1}{x}} = \infty$$

Eg: Let X be a discrete

RV with support $\mathbb{N}$
and PMF

$$p(x) = \begin{cases} \frac{1}{cx^3} & \text{if } x \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$

where

$$c = \sum_{i=1}^{\infty} \frac{1}{x^3} < \infty$$

Mean,

$$E(X) = \sum_{x=1}^{\infty} x \cdot \frac{1}{cx^3}$$

$$= \frac{1}{c} \sum_{x=1}^{\infty} \frac{1}{x^2}$$

$$= \frac{1}{c} \zeta(2) < \infty$$

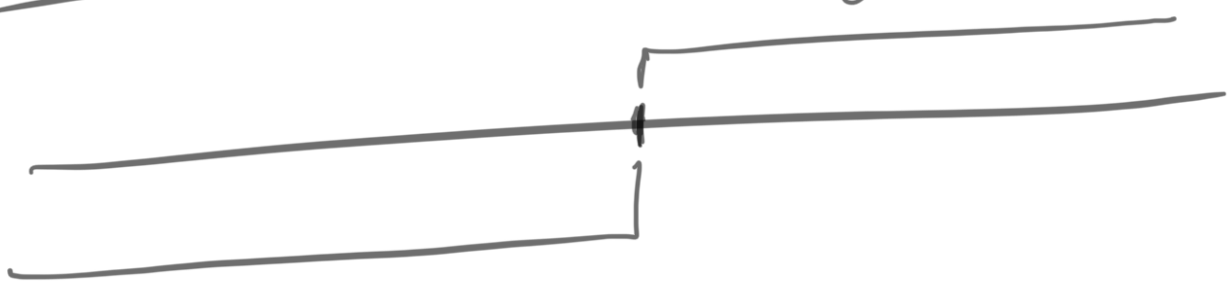
Second moment,

$$E(X^2) = \sum^{\infty} x^2 \cdot \frac{1}{\ldots}$$

$E(X^2) = \sum_{x=1} \frac{}{c\,x^3}$

$$= \frac{1}{c} \sum_{x=1}^{\infty} \frac{1}{x} = \infty$$

Median:



Def (Degenerate RV)

Let X be a random variable with PMF

$$\mathbb{P}(X=a) = 1 \text{ for an } a \in \mathbb{R}$$

said to be

Then X is said to be a degenerate RV at a.

Lemma. Let X be a degenerate RV at a. Then

(i) $\mathbb{E}(X) = a$

(ii) $Var(X) = 0$

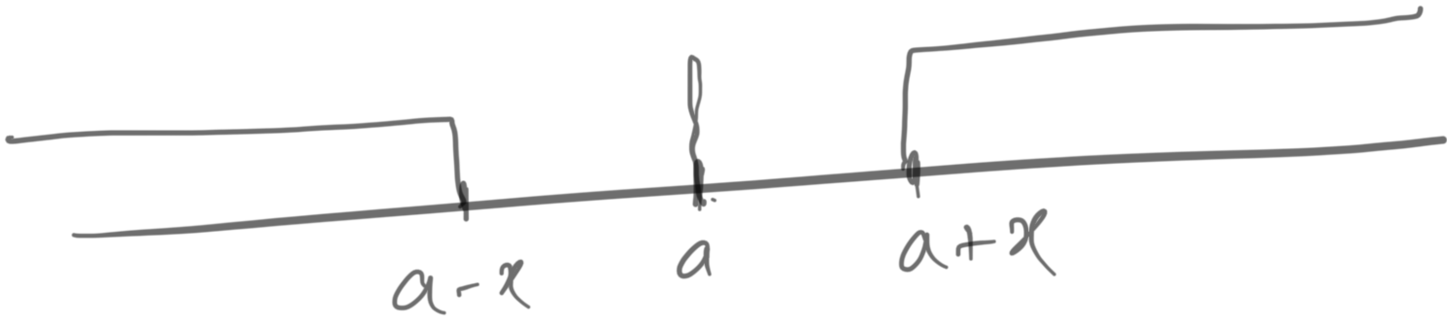
Symmetric Random Variable

Def. A random variable X is said to be a symmetric RV with respect to $a \in \mathbb{R}$, if

~~for~~ all $x > 0$

7°

$$\mathbb{P}(X \leq a - x) = \mathbb{P}(X \geq a + x)$$



Usually symmetric RV refers to symmetric about .