Feb 10

Def: X is a multivariate random vector.

i.e. $X = (X_1, \ldots, X_n)^T$

$$E(X) = \begin{bmatrix} E(X_1) \\ E(X_2) \\ \vdots \\ E(X_n) \end{bmatrix}$$

Def: Let $(X_1, X_2, X_3)^T$ is a trivariate RV.

The covariance matrix of $(X_1, X_2, X_3)^T$ is defined as

$$
\begin{array}{c|ccc}
 & x_1 & x_2 & x_3 \\
\hline
x_1 & \text{Cov}(x_1, x_1) & \text{Cov}(x_1, x_2) & \text{Cov}(x_1, x_3) \\
x_2 & \text{Cov}(x_2, x_1) & \text{Cov}(x_2, x_2) & \text{Cov}(x_2, x_3) \\
x_3 & \text{Cor}(x_3, x_1) & \text{Cov}(x_3, x_2) & \text{Cov}(x_3, x_3)
\end{array}
$$

$$
= \begin{pmatrix}
\text{Var}(x_1) & \text{Cor}(x_1, x_2) & \text{Cor}(x_1, x_3) \\
\text{Cov}(x_1, x_2) & \text{Var}(x_2) & \text{Cov}(x_2, x_3) \\
\text{Cov}(x_1, x_3) & \text{Cov}(x_2, x_3) & \text{Var}(x_3)
\end{pmatrix}
$$

Let $X = (x_1, \ldots, x_n)$

Then $\text{Cov}(X)$

$$
= \begin{pmatrix}
\sigma_{11} & \sigma_{12} & \cdots & \sigma_{1n} \\
 & & & \sigma_{2n}
\end{pmatrix}
$$

$$\begin{pmatrix} \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2n} \\ \vdots & & & \\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_{nn} \end{pmatrix}$$

where

$$\sigma_{ii} = Var(X_i)$$

$$\sigma_{ij} = cov(X_i, X_j)$$

Multivariate Normal (Gaussian) RV

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let $\underset{\sim}{X}$ be a p-variate RV with

$$E(\underset{\sim}{X}) = E\begin{bmatrix} X_1 \\ \vdots \\ X_p \end{bmatrix} = \begin{bmatrix} E(X_1) \\ \vdots \\ E(X_p) \end{bmatrix} = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_p \end{pmatrix}$$

$= \underset{\sim}{\mu}$

$Cov(\underset{\sim}{X}) = \underset{\sim}{\Sigma}$

if $\underset{\sim}{X}$ has joint PDF

$$f_{\underset{\sim}{X}}(\underline{x}) = \frac{1}{(2\pi)^{p/2} |\Sigma|^{1/2}} e^{-\frac{1}{2}(\underline{x}-\underset{\sim}{\mu})^T \underset{\sim}{\Sigma}^{-1} (\underset{\sim}{x}-\underset{\sim}{\mu})}$$

for $\underline{x} \in \mathbb{R}^p$

then $\underset{\sim}{X}$ is said to follow p-variate Gaussian distribution ... μ

with parameters $\underset{\sim}{\mu}$ & $\underset{\sim}{\Sigma}$. This is denoted by $\underset{\sim}{X} \sim N_p(\underset{\sim}{\mu}, \underset{\sim}{\Sigma})$

If $p = 1$, the PDF reduces to

$$f_X(x) = \frac{1}{(2\pi)^{\frac{1}{2}}(\sigma^2)^{1/2}} e^{-\frac{1}{2}(x-\mu)(\sigma^2)^{-1}(x-\mu)}$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

This is univariate $N(\mu, \sigma^2)$ PDF.

D... ... ... matrix

Properties of cov matrix

(i) It is a symmetric matrix

(ii) It is a positive semidefinite matrix.
(without proof)

Hint: (i) obvious

Lemma: If $\underset{\sim}{X}$ is a RV with $\mathbb{E}(\underset{\sim}{X}) = \underset{\sim}{\mu}$ and $Cov(\underset{\sim}{X}) = \Sigma$, then for $\underset{\sim}{A} \in \mathbb{R}^{p \times p}$ & $\underset{\sim}{b} \in \mathbb{R}^{p}$

(i) $\mathbb{E}(\underset{\sim}{A}\underset{\sim}{X} + \underset{\sim}{b}) = \underset{\sim}{A}\underset{\sim}{\mu} + \underset{\sim}{b}$

(ii) $\mathrm{Cov}(\underset{\sim}{A}\underset{\sim}{X} + \underset{\sim}{b})$

$$= \underset{\sim}{A}\,\underset{\sim}{\Sigma}\,\underset{\sim}{A}^{T}$$

~~without proof~~

<u>Lemma</u>: For bivariate normal RV, zero covariance implies independence of components.

Hint:

$$\mathrm{Cov}(\underset{\sim}{X}) = \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{pmatrix}$$

$$f_{\underset{\sim}{X}}(x_1, x_2) \qquad -\frac{1}{2}\left(\frac{x_1 - \mu_1}{\sigma_1}\right)^2$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_1} e^{-} \quad \frac{1}{\sqrt{2\pi}\,\sigma_2} e^{-\frac{1}{2}\left(\frac{x_2-\mu_2}{\sigma_2}\right)^2}$$

Identically distributed RVs

Two RVs X and Y are said to be identically distributed if they have same CDF.

(i) If X and Y are discrete then identically distributed requires same PMF

(ii) . . . - - continuous

PDF

Notation $X \overset{d}{=} Y$

Independently and identically distributed (IID)

Def: A set of RVs $X_1, \dots, X_n$ are said to be IID if

(I) $X_i \overset{d}{=} X_j \quad \forall i \neq j$

(II) $(X_1, \dots, X_n)$ are independence.

Theorem: Let $X_1, \dots, X_n$ are IID RVs with mean $\mu = E(X_1)$ and variance $\sigma^2 = Var(X_1)$.

Define
$$\bar{X}_n = \frac{X_1 + \dots + X_n}{n},$$
it is known as sample mean.

Then,

(i) $E(\bar{X}_n) = \mu$

(ii) $Var(\bar{X}_n) = \frac{\sigma^2}{n}$

Moment Generating Function

(MGF)

Def: let X be a RV. The MGF of X is denoted by $M_X(t)$, $t \in \mathbb{R}$, and defined as

$$M_X(t) = E\left(e^{tX}\right)$$

$$E\left(e^{tX}\right) = 1 + t\,E(X) + \frac{t^2}{2!}E(X^2) + \ldots$$