Measurable space:

Ω is a non empty set

$\mathcal{F}$ is a σ-algebra/field on Ω

Then $(\Omega, \mathcal{F})$ is known as measurable space.

Probability measure:

Let $(\Omega, \mathcal{F})$ be a measurable space. Suppose $\mathbb{P} : \mathcal{F} \to [0,1]$ is a set function satisfying following:

(1) $\mathbb{P}(\Omega) = 1$ or $\mathbb{P}(\phi) = 0$

(2) If $\{A_i\}_{i}^{\infty} \subset \mathcal{F}$ and

(ii) if $\{A_i\}_{i=1}^{\infty}$

$A_i \cap A_j = \phi$ if $i \neq j$

(A_i's are pairwise disjoint)

then $$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$$

Then $\mathbb{P}$ is said to be a probability measure on $(\Omega, \mathcal{E})$.

Remark: $(\Omega, \mathcal{E}, \mathbb{P})$ is said to be a probability space.

$\phi =$

Axioms of probability:

(i) Non-negativity:

For any $A \in \mathcal{F}$,

$$\mathbb{P}(A) \geq 0$$

(ii) Totality:

$$\mathbb{P}(\Omega) = 1$$

(iii) Sigma-additivity:

If $A_i \in \mathcal{F}$ for $i = 1, 2, \dots$

and $A_i \cap A_j = \phi \ \forall i \neq j$, then $\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$

Lemma: If a measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$ satisfy axioms of probability then it is a valid probability measure on $(\Omega, \mathcal{F})$.

In other words $(\Omega, \mathcal{F}, \mathbb{P})$ is a valid probability space.

Eg: Tossing of two coins

g. Tossing of

$\Omega = \{HH, HT, TH, TT\}$

$\mathcal{F} = 2^{\Omega}$

Assume equally likely case.

That is

$\mathbb{P}(\{\omega\}) = \frac{1}{4}$ if $\omega \in \Omega$

and for any $A \in \mathcal{F}$

$$\mathbb{P}(A) = \sum_{\omega \in A} \mathbb{P}(\{\omega\})$$

Then,

(1) $$\mathbb{P}(\Omega) = \sum_{\omega \in \Omega} P(\{\omega\})$$

$$= \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4}$$

$$= 1$$

(ii) $\mathbb{P}(A) \geq 0$

(iii) By def. of $\mathbb{P}$.

Lemma: Naive prob. measure on Ω is a probability measure on $(\Omega, 2^{\Omega})$.

Eg: Same Ω & $\mathcal{F}$ as previous example.

Define
$$\mathbb{P}(\{\omega_i\}) = p_i \quad , i = 1, 2, 3, 4$$
$$p_i \geq 0 \text{ and } \sum_{i=1}^{4} p_i = 1$$

$$\mathbb{P}(A) = \sum_{\omega \in A} \mathbb{P}(\{\omega_i\})$$

where $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$

Then, $\mathbb{P}$ is a prob. measure on $(\Omega, \mathcal{G})$

Discrete probability space

Ω is countable

finite

infinite

Def: let Ω is countable and $\mathcal{G}$ be a σ-algebra on Ω. (e.g. power set is a sigma-algebra).

If $\mathbb{P}$ is a probability measure on $(\Omega, \mathcal{F})$, then

(i) $\mathbb{P}$ is known as a discrete probability measure on $(\Omega, \mathcal{F})$

(ii) $(\Omega, \mathcal{F}, \mathbb{P})$ is known as a discrete probability space.

Eg: $\Omega = \mathbb{N}$

$\mathcal{F} = 2^{\mathbb{N}}$

$$\mathbb{P}(\{i\}) = \begin{cases} \frac{1}{2^i} & \text{if } i \in \mathbb{N} \\ 0 & \text{o/w} \end{cases}$$

For $A \subseteq \mathbb{N}$

$$\mathbb{P}(A) = \sum_{i \in A} \mathbb{P}(\{i\})$$

Axiom

(I) $\mathbb{P}(A) \geq 0$

(II) $\mathbb{P}(\Omega) = \sum_{i=1}^{\infty} \frac{1}{2^i} = 1$

(III) true by assumption.

Lemma: For a countable sample space Ω and a σ-algebra $\mathcal{F}$ on Ω, to define a probability measure:

it is enough to define probability on singleton sets and use third axiom of probability to extend it to all other sets.

Borel σ-algebra:

eg.1 $\Omega = \{H, T\}$

$Q = \{\{H\}\}$

we want to construct a σ-algebra using Q

$\{\phi, \Omega, \{H\}, \{T\}\}$

eg 2

$\Omega = \{a, b, c\}$

$\mathcal{C} = \{\{a, b\}\}$

A σ-field contang $\mathcal{C}$ is

$\{\phi, \Omega, \{a, b\}, \{c\}\}$

Minimal σ-field: Let $\mathcal{C}$ be a collection of subsets of a non empty set Ω.

If $\mathcal{C}_i$'s are σ-field containing $\mathcal{C}$, then

minimal σ-field generated

minimal [illegible]
by Q is defined as

$$\tau_Q = \bigcap_i \tau_i$$
