Jan 27

Lemma: Let X be a continuous random variable with CDF F and pdf f. Then, for all $x \in \mathbb{R}$

$$f(x) = \frac{d}{dx} F(x)$$

Recall
$$F(x) = \int_{-\infty}^{x} f(t)\, dt$$

Theorem: (Linearity of expectation)
Let X be a RV and $g(\cdot)$ and $h(\cdot)$ are two

function on $\mathbb{R} \to \mathbb{R}$, and

$a, b \in \mathbb{R}$. Then

$$\mathbb{E}[a\, g(X) + b\, h(X)]$$

$$= a\, \mathbb{E}[g(X)] + b\, \mathbb{E}[h(X)].$$

Pf: Case I

When X is a discrete RV.

Let $p(\cdot)$ be the PMF of X with support

$$S = \{a_1, a_2, \ldots\}.$$

Now, $\mathbb{E}[a\, g(X) + b\, h(X)]$

$$= \sum [a\, g(x) + b\, h(x)]\, p(x)$$

$x \in S$

$$= \sum_{x \in S} \left[a\, g(x) \cdot p(x) + b\, h(x) \cdot p(x) \right]$$

$$= \sum_{x \in S} a\, g(x)\, p(x) + \sum_{x \in S} b\, h(x) \cdot p(x)$$

$$= a \sum_{x \in S} g(x)\, p(x) + b \sum_{x \in S} h(x)\, p(x)$$

$$= a\, \mathbb{E}[g(X)] + b\, \mathbb{E}[h(X)].$$

Theorem: Let X be a RV. Suppose $g_1, g_2, \ldots, g_k$ be functions $\mathbb{R} \to \mathbb{R}$ and $a_1, \ldots, a_k \in \mathbb{R}$.

Then

$$\mathbb{E}\left[\sum_{i=1}^{k} a_i\, g_i(X)\right] = \sum_{i=1}^{k} a_i\, \mathbb{E}\left[g_i(X)\right]$$

Moments:

Def: $r \in \mathbb{N}$, X is a RV then r^{th} moment is defined as $\mathbb{E}(X^r)$.

Mean of a RV:

Def: let X be a RV. its mean is defined as $\mathbb{E}(X)$.

Raw moments:

r^{th} raw moment is defined as $\mathbb{E}(X^r)$

Central moments:

r^{th} central moment is defined as

$$\mathbb{E}\left[\left(X - \mathbb{E}(X)\right)^r\right]$$

Lemma $\quad \mathbb{E}\left(X - \mathbb{E}(X)\right) = 0$

Pf: let X is discrete

$$\mathbb{E}\left(X - \underbrace{\mathbb{E}(X)}_{\text{constant}}\right)$$

let $\mu = \mathbb{E}(X)$

Now

$$\mathbb{E}(X - \mu)$$

$$= \sum_{x \in S} (x - \mu)\, p(x)$$

$$\sum \mu\, p(x)$$

$$= \sum_{x \in S} x\, p(x) - \sum_{x \in S} \mu\, p(x)$$

$$= \mathbb{E}(X) - \mu \underbrace{\sum_{x \in S} p(x)}$$

$$= \mathbb{E}(X) - \mathbb{E}(X)$$

$$= 0$$

Measure of Spread

$$\mathbb{E}\underline{(X - \mathbb{E}(X))} = 0$$

Eg: let X, is RV with support $\{99, 100, 101\}$

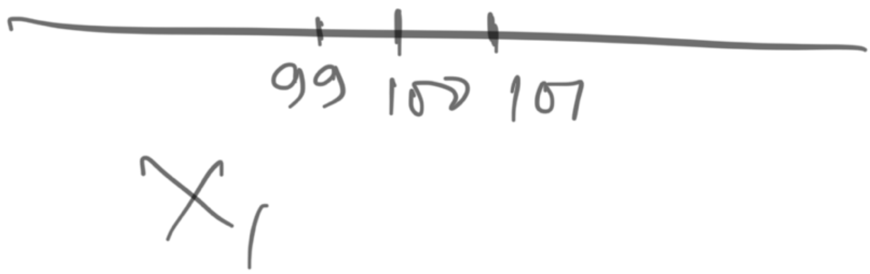
prob. law - equally likely

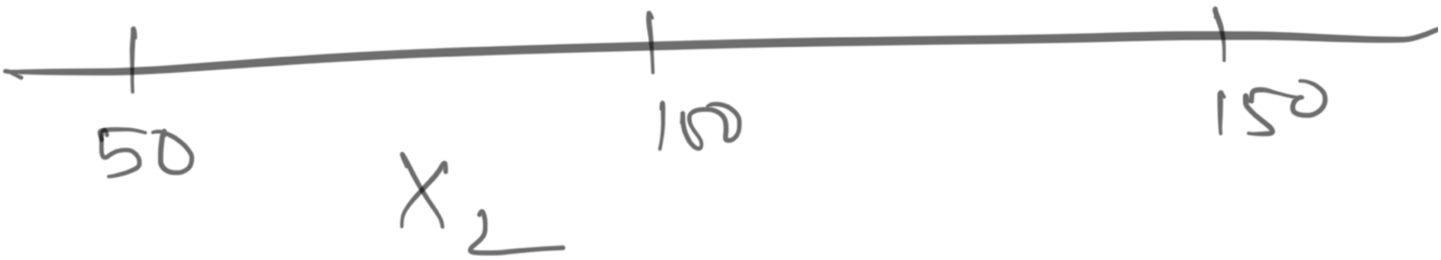
let X_2 be RV with support $\{50, 100, 150\}$

prob. law – equally likely.

observe that

$$E(X_1) = E(X_2)$$





A suitable measure of spreadi

$$E[(X - E(X))]$$

This is known as variance

A RV X.

Def:

$$Var(X) = E[(X - E(X))^2]$$

$\boxed{\text{Lemma: Let } X \text{ be a RV. Then } Var(X) = E(X^2) - [E(X)]^2}$

proof: $Var(X)$

$$= E[(X - E(X))^2]$$

$$= E[X^2 + [E(X)]^2 - 2X E(X)]$$

linearity

$$E[X^2] + E[(E(X))^2]$$

$= E[X^2] + E[\underbrace{E(X)^2}_{\text{constant}}]$

$- E[2 X E(X)]$

$= E(X^2) + [E(X)]^2$

$- 2E(X) \; E(X)$

$= E(X^2) - [E(X)]^2$

Lemma: Let X be a RV.

Then

(i) $E(a) = a$

for all fixed a

(ii) $E(aX) = aE(X)$

for any fixed a

Eg: Let X be a Bernoulli RV with parameter p.

Then
$$\mathbb{P}(X = x) = \begin{cases} p & \text{if } x = 1 \\ 1-p & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbb{E}(X) = p$$

$$\mathbb{E}(X^2) = \sum_{x \in S} x^2 \cdot p(x)$$

$$= 1^2 \cdot p + 0^2 \cdot (1-p)$$

$$= p$$

$$\mathrm{Var}(X) = \mathbb{E}(X^2) - [\mathbb{E}(X)]^2$$

$$= p - p^2$$

$= p(1-p)$

Remark: ① For any RV X,

$$Var(X) \geq 0$$

② Working definition

$$Var(X) = E(X^2) - [E(X)]^2$$

Lemma:

$$E(X^2) \geq [E(X)]^2$$

Standard discrete distribution

① Bernoulli distribution

Def: X is a said to follow ... with parameter p if

Bernoulli distribution if its PMF is given by

$$P(X=x) = \begin{cases} p & \text{if } x=1 \\ 1-p & \text{if } x=0 \\ 0 & \text{otherwise} \end{cases}$$

Result (i) $E(X) = p$
(ii) $var(X) = p(1-p)$

Notation
$X \sim \text{Bernoulli}(p)$

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② Binomial distribution

Def: X is said to follow Binomial distribution with parameters (n, p), where $n \in \mathbb{N}$, $p \in (0,1)$, if its PMF is given by if its pmf is

$$P(X=x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x} & \text{if } x = 0, 1, 2, \dots, n \\ 0 & \text{, otherwise} \end{cases}$$

Notation: $X \sim \text{Binomial}(n, p)$

Lemma: Let $X \sim \text{Binomial}(n, p)$.

Then, (i) $E(X) = np$

(ii) $Var(X) = np(1-p)$

proof: (i) $E(X)$

$$= \sum_{x \in \{0, 1, 2, \dots, n\}} x \cdot p(x)$$

$$= \sum_{x=0}^{n} x \cdot \binom{n}{x} \cdot p^x (1-p)^{n-x}$$

$$x=0$$

$$= \sum_{x=0}^{n} x \cdot \frac{n!}{x!\,(n-x)!}\, p^{x} \cdot (1-p)^{n-x}$$

$$= \sum_{x=1}^{n} \cancel{x} \cdot \frac{n!}{\cancel{x}\,(x-1)!\,(n-x)!}\, p^{x}\,(1-p)^{n-x}$$

$$= \sum_{x=1}^{n} \frac{n\,(n-1)!}{(x-1)!\,((n-1)-(x-1))!}\, p \cdot p^{x-1}\,(1-p)^{n-1-(x-1)}$$

$$= np \sum_{x=1}^{n} \frac{(n-1)!}{(\underbrace{x-1}_{r})!\,(n-1-(x-1))!}\, p^{x-1}\,(1-p)^{n-1-(x-1)}$$

$$= np \underbrace{\sum_{r=0}^{n-1} \frac{(n-1)!}{r!\,(n-1-r)!}\, p^{r}\,(1-p)^{n-1-r}}$$

$$\left[\, p + (1-p) \,\right]^{n-1}$$

$= np \; [p + (1-p)]$

$= np$

(ii) Var — do yourself

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(3) Poisson distribution

X is said to be a Poisson RV with parameter λ, $\lambda > 0$, if its PMF is given by

$$P(X=x) = \begin{cases} \dfrac{e^{-\lambda}\lambda^{x}}{x!}, & \text{if } x = 0, 1, 2, \dots \\ 0 & \text{otherwise} \end{cases}$$

Notation

$X \sim \text{Poisson}(\lambda)$.

Lemma: If $X \sim \text{Poisson}(\lambda)$

then (i) $\mathbb{E}(X) = \lambda$

(ii) $\text{Var}(X) = \lambda$

Proof: (i) $\mathbb{E}(X)$

$$= \sum_{x=0}^{\infty} x \cdot \frac{e^{-\lambda} \lambda^x}{x!}$$

$$= \sum_{x=1}^{\infty} x \cdot \frac{e^{-\lambda} \lambda^x}{x!}$$

$$= \sum_{x=1}^{\infty} \cancel{x} \cdot \frac{e^{-\lambda} \lambda^{x-1} \cdot \lambda}{\cancel{x} \cdot (x-1)!}$$

$$= \lambda \sum^{\infty} \frac{e^{-\lambda} \lambda^{x-1}}{}$$

$$x=1 \qquad \underbrace{(x-1)}_{r}!$$

$$= \lambda \left(\sum_{r=0}^{\infty} \frac{e^{-\lambda} \lambda^{r}}{r!} \right)$$

$$= \lambda$$