MTL108: Problem Set-2

IIT Delhi

Problem 1 There are k people in a room. Assume each person's birthday is equally likely to be any of the 365 days of the year (we exclude February 29), and that people's birthdays are independent (we assume there are no twins in the room). What is the probability that two or more people in the group have the same birthday?

Problem 2 Suppose we choose a positive integer at random, according to some unknown probability distribution. Suppose we know that $\mathbb{P}(\{1,2,3,4,5\}) = 0.3$, $\mathbb{P}(\{4,5,6\}) = 0.4$ and $\mathbb{P}(\{1\}) = 0.1$. What are the largest and smallest possible values of $\mathbb{P}(\{2\})$?

Problem 3 Show that for any three events A, B and C,

$$\mathbb{P}(A\cup B\cup C) = \mathbb{P}(A)+\mathbb{P}(B)+\mathbb{P}(C)-\mathbb{P}(A\cap B)-\mathbb{P}(A\cap C)-\mathbb{P}(B\cap B)+\mathbb{P}(A\cap B\cap C).$$

Problem 4 (*Inclusion-Exclusion Principle for n Events*). Let $A_1, A_2, \dots, A_n$ be n events in a probability space (Ω, Σ, P). The probability of their union is given by the inclusion-exclusion principle:

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i) - \sum_{1\le i<j\le n} P(A_i\cap A_j) + \sum_{1\le i<j<k\le n} P(A_i\cap A_j\cap A_k) - \cdots + (-1)^{n+1} P\left(\cap_{i=1}^{n} A_i\right),$$

where the terms alternate in sign, with the k-th sum (for $k = 1$ to n) including all intersections of k distinct events.

Problem 5 (*Boole's Inequality*) Let $A_1, A_2, \dots, A_n$ be n events in a probability space (Ω, Σ, P). Then, probability of the union of these events satisfies:

$$P\left(\bigcup_{i=1}^{n} A_i\right) \le \sum_{i=1}^{n} P(A_i).$$

Problem 6 Two fair dice are rolled. Let X and Y be the outcome of the first die and the second die, respectively. Then, which of the following statement(s) is (are) true.

(a) $P(X+Y = \text{ a prime number }) = \frac{5}{12}$,

(b) $P(|X-Y| = \text{ a prime number }) = \frac{4}{9}$,

(c) $P(X+Y = \text{ a perfect square}) = \frac{1}{6}$,

(d) $P(|X-Y| = \text{ a prime number }) = \dfrac{5}{9}$.

Problem 7 Three biscuit making machines A, B and C are installed in a factory. Machine A makes 35% of the biscuits, machine B makes 27% of the biscuits and rest of the biscuits are made by machine C. It is found that 4% of the biscuits made by machine A are broken, 1% of the biscuits made by machine B are broken and 9% of the biscuits made by machine C are broken. Select a biscuit at random. Given that the biscuit is broken, what is the probability that it is NOT made by machine A?

Problem 8 A fair die is independently rolled two times. Let X and Y be outcomes on the dice. Define $Z = X + Y$ and W to be the remainder obtained when Z is divided (integer division) by 6. Then, **prove or disprove** that

(a) Events $\{X = a\}$ and $\{W = b\}$ are independent for $a = 4, 5, 6$ and $b = 0, 1, 2$,

(b) Events $\{X = a\}$ and $\{W = b\}$ are independent for $a = 1$, $b = 5$,

(c) Events $\{X = a\}$ and $\{Z = b\}$ are independent for $a = 1$, $b = 1$,

(d) Events $\{X = a\}$ and $\{Z = b\}$ are independent for $a = 1$, $b = 5$.

Problem 9 Let A and B be two events such that $P(A) > 0$ then prove or disprove that

$$P(B|A) \geq 1 + \frac{P(B)}{P(A)} - \frac{1}{P(A)}.$$