March 18

① $X \sim \chi^2_{k_1}$

② $Y \sim \chi^2_{k_2}$

③ X and Y are indep.

then

$$M_{X+Y}(t) = M_X(t) \cdot M_Y(t)$$

$$= (1-2t)^{-k_1/2} (1-2t)^{-k_2/2}$$

$$= \underbrace{(1-2t)^{-(k_1+k_2)/2}}_{\text{MGF of } \chi^2_{k_1+k_2}}$$

By characterization of MGF

we have

$$X+Y \sim \chi^2_{k_1+k_2}$$

$X_1, \dots, X_n$ IID $N(\mu, \sigma^2)$

$$\bar{X} = \frac{X_1 + \dots + X_n}{n}$$

$$D_1 = X_1 - \bar{X}$$

$$\mathrm{Cov}(\bar{X}, D_1)$$

$$= \mathrm{Cov}(D_1, \bar{X})$$

$$= \mathrm{Cov}(X_1 - \bar{X}, \bar{X})$$

$=$

Property

$\mathrm{Cov}(X+Y, Z)$

$$\mathrm{Cov}(X + Y, Z) = \mathrm{Cov}(X, Z) + \mathrm{Cov}(Y, Z)$$

$$= \mathrm{Cov}(X_1, \bar{X}) - \mathrm{Cov}(\bar{X}, \bar{X})$$

$$= \underbrace{\mathrm{Cov}(X_1, \bar{X})}_{} - \frac{\sigma^2}{n}$$

$$= \frac{\sigma^2}{n} - \frac{\sigma^2}{n} = 0$$

why?

$$\mathrm{Cov}(X_1, \bar{X})$$

$$= \mathrm{Cov}\left(X_1, \frac{1}{n}(X_1 + \dots + X_n)\right)$$

$$= \mathrm{Cov}\left(X_1, \frac{1}{n}X_1\right)$$

$$+ \dots$$

$$+ \operatorname{cov}\left(X_1, \frac{1}{n} X_2\right) + \cdots$$

$$+ \operatorname{cov}\left(X_1, \frac{1}{n} X_n\right)$$

$$= \frac{1}{n} \left[\operatorname{cov}(X_1, X_1) + \operatorname{cov}(X_1, X_2) + \cdots + \operatorname{cov}(X_1, X_n) \right]$$

$$= \frac{1}{n} \left[\sigma^2 + 0 + \cdots + 0 \right]$$

$$= \frac{\sigma^2}{n}$$

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property: If (X, Y) is jointly normally distributed and $\operatorname{cov}(X, Y) = 0$

then X and Y are independent.

$\bar{X}$ is independent of $(D_1, \dots, D_n)$

$\Rightarrow \bar{X}$ is independent of $g(D_1, \dots, D_n)$

where $g: \mathbb{R}^n \to \mathbb{R}^p$

Take

$$g(D_1, \dots, D_n) = \frac{1}{n-1} \sum_{i=1}^{n} D_i^2 = S^2$$

the nym

$\bar{X}$ is independent of S^2.

$X_1, \ldots X_n$ are IID $N(0,1)$ RVs.

We know

(i) $\bar{X}$ and S^2 are independent

(ii) $\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$

(iii) $D_i \sim N(0, ??)$

Sketch of proof:

$\frac{X_i - \mu}{}$

$$Y_i = \frac{\quad}{\sigma}$$

Then $Y_i \overset{IID}{\sim} N(0,1)$

(i) Use definition of χ^2

$$\sum_{i=1}^{n} Y_i^2 \sim \chi^2_n$$

(ii) $$\sum_{i=1}^{n} Y_i^2 = \frac{1}{\sigma^2}\sum_{i=1}^{n} (X_i - \mu)^2$$

$$= \frac{1}{\sigma^2}\sum_{i=1}^{n} (\underbrace{X_i - \bar{X}} + \underbrace{\bar{X} - \mu})^2$$

$$= \frac{1}{\sigma^2}\sum_{i=1}^{n} \Big[(X_i - \bar{X})^2 + (\bar{X} - \mu)^2 + 2(X_i - \bar{X})\cdot(\bar{X}-\mu) \Big]$$

$$= \frac{1}{\sigma^2} \sum_{i=1} (x_i - x)$$

$$+ \frac{1}{\sigma^2} \sum_{i=1}^{n} (\bar{x} - \mu)^2$$

$$+ \frac{2}{\sigma^2} \sum_{i=1}^{n} (x_i - \bar{x}) \cdot (\bar{x} - \mu)$$

IIIrd term

$$\frac{2}{\sigma^2} (\bar{x} - \mu) \sum_{i=1}^{n} (x_i - \bar{x})$$

$$= 0$$

$$= \frac{1}{\sigma^2} \sum_{i=1}^{n} D_i^2 + \frac{n}{\sigma^2} (\bar{x} - \mu)^2$$

$$= \frac{(n-1) s^2}{\sigma^2} + \left(\frac{\bar{x} - \mu}{\sigma / \sqrt{n}} \right)^2$$

Since $\bar{X} \sim N(\mu, \sigma^2/n)$

$$\Rightarrow \frac{\bar{X}-\mu}{\sqrt{\sigma^2/n}} \sim N(0,1)$$

$$\Rightarrow \left(\frac{\bar{X}-\mu}{\sqrt{\sigma^2/n}}\right)^2 \sim \chi^2_1$$

$$\sum_{i=1}^{n} Y_i^2 \sim \chi^2_n$$

$$= \frac{(n-1)S^2}{\sigma^2} + \underbrace{\left(\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}\right)^2}_{\chi^2_1}$$

independent

$$= \underbrace{W + \chi^2_1}_{\text{independent}}$$

Therefore

$$W = \frac{(n-1)s^2}{\sigma^2} \sim \chi^2_{n-1}$$