MTL108: Problem Set-1

IIT Delhi

Problem 1: A box contains 3 identical red balls and 2 identical blue balls. Two balls are drawn without replacement. A student uses naive probability, assuming all pairs are equally likely (ignoring order), and calculates the probability of getting two red balls as $\frac{3}{5} \cdot \frac{2}{4}$. Compute the probability using the combinatorial approach and compare with the student's approach.

Problem 2: In a survey, 30 students reported whether they liked their tea, coffee, or soft-drink. 15 liked tea, 20 liked coffee, and 9 liked soft-drink. Additionally, 12 students liked both tea and coffee, 5 liked coffee and soft-drink, 6 liked tea and soft-drink, and 3 liked all three. How many students dislike all three? Explain why your answer is correct.

Problem 3: Consider a biased coin with sample space $\Omega = \{H, T\}$ and σ-algebra $\mathcal{C} = 2^{\Omega}$, the power set of Ω. The probability measure is defined as $P(\{H\}) = 0.7$, $P(\{T\}) = 0.3$, and for any $A \in 2^{\Omega}$, $P(A) = \sum_{\omega \in \Omega} P(\{\omega\})$. Verify that P satisfies the axioms of a probability measure. Then, compute the probability of the event $\{H, T\}$.

Problem 4: Two fair six-sided dice are rolled. Define the probability space $(\Omega, \mathcal{C}, P)$, where Ω is the set of all ordered pairs (i, j), $i, j \in \{1, 2, \ldots, 6\}$. Specify $\mathcal{C}$ as the power set and define P. Calculate the probability of the event $A = \{(i, j) \mid i+j \geq 10\}$.

Problem 5: Let $\Omega = \{1, 2, 3, 4\}$ and $\mathcal{C} = \{\emptyset, \{1, 2\}, \{3, 4\}, \{1, 2, 3, 4\}\}$. Check if $\mathcal{C}$ is a σ-algebra by testing all three axioms. If it fails, identify which axiom is violated and construct a minimal σ-algebra containing $\{1, 2\}$.

Problem 6: In a naive attempt to find the probability that a randomly chosen point in a unit square $[0, 1] \times [0, 1]$ lies within the circle inscribed in it (radius 0.5, center $(0.5, 0.5)$), a student assumes points are equally likely and estimates the probability as the ratio of diameters. Do you agree with his/her approach? Explain your reasoning in either case.

Problem 7: A probability space has $\Omega = \{a, b, c, d\}$, $\mathcal{C} = 2^{\Omega}$, and probability measure $P(\{a\}) = 0.2$, $P(\{b\}) = 0.3$, $P(\{c\}) = 0.4$, $P(\{d\}) = 0.1$. Compute the probabilities of the events $A = \{a, b\}$, $B = \{b, c\}$, and their union $A \cup B$. Verify that the probability measure satisfies $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Problem 8: Show that for any two events A and B,

$$\mathbb{P}(A) + \mathbb{P}(B) - 1 \leq \mathbb{P}(A \cap B) \leq \mathbb{P}(A \cup B) \leq \mathbb{P}(A) + \mathbb{P}(B).$$

For each of these three inequalities, give a simple criterion for when the inequality is actually an equality (e.g., give a simple condition such that $\mathbb{P}(A \cap B) = \mathbb{P}(A \cup B)$ if the condition holds).

Problem 9: Prove that an algebra may not be a σ-algebra.