MTL108: Solution to Problem Set-9

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Problem 1

Determine the maximum likelihood estimator of θ for the density function $f(x) = e^{-(x-\theta)}$ for $x \geq \theta$, and 0 otherwise.

Solution:
For a sample $X_1, X_2, \ldots, X_n$, the likelihood function $L(\theta)$ is the product of the individual density functions. Because the density is 0 when $x < \theta$, the likelihood is only non-zero if all observed values are greater than or equal to θ. We can represent this using an indicator function $I(\min(x_i) \geq \theta)$:

$$L(\theta) = \prod_{i=1}^{n} e^{-(x_i-\theta)} \cdot I(x_i \geq \theta)$$

$$L(\theta) = e^{-\sum_{i=1}^{n} x_i + n\theta} \cdot I(\min(x_i) \geq \theta)$$

To find the Maximum Likelihood Estimator (MLE), we want to maximize $L(\theta)$. Looking at the exponential term $e^{-\sum x_i + n\theta}$, since $n > 0$, this function is strictly increasing with respect to θ. Therefore, to maximize the likelihood, we must make θ as large as possible.

However, the indicator function restricts θ from being larger than the smallest observation in our sample. Thus, the largest possible valid value for θ is the minimum order statistic.

Final Answer:

$$\hat{\theta} = \min(X_1, X_2, \ldots, X_n) = X_{(1)}$$

Problem 2

Determine the MLE of θ for the Laplace density function $f(x) = \frac{1}{2}e^{-|x-\theta|}$ for $-\infty < x < \infty$.

Solution:
The likelihood function for a sample of size n is:

$$L(\theta) = \prod_{i=1}^{n} \frac{1}{2} e^{-|x_i-\theta|} = \left(\frac{1}{2}\right)^n e^{-\sum_{i=1}^{n} |x_i-\theta|}$$

We take the natural logarithm to obtain the log-likelihood function:

$$\ell(\theta) = -n\ln(2) - \sum_{i=1}^{n} |x_i - \theta|$$

To maximize $\ell(\theta)$, we must minimize the subtracted sum:

$$\text{Minimize} \sum_{i=1}^{n} |x_i - \theta|$$

It is a well-known mathematical property that the sum of absolute deviations is minimized when θ is the sample median.

Final Answer:

$$\hat{\theta} = \text{Median}(X_1, X_2, \ldots, X_n)$$

Problem 3

Let $X_1, \ldots, X_n$ be a sample from a $N(\mu, \sigma^2)$ population. Determine the MLE of σ^2 when μ is known. What is the expected value of this estimator?

Solution:

Because μ is known, it acts as a constant, not a parameter to be estimated. The likelihood function is:

$$L(\sigma^2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x_i - \mu)^2}{2\sigma^2}\right) = (2\pi\sigma^2)^{-n/2} \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \mu)^2\right)$$

The log-likelihood function (letting $v = \sigma^2$ for easier derivation) is:

$$\ell(v) = -\frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln(v) - \frac{1}{2v}\sum_{i=1}^{n}(x_i - \mu)^2$$

Take the derivative with respect to v and set it to zero:

$$\frac{\partial \ell}{\partial v} = -\frac{n}{2v} + \frac{1}{2v^2}\sum_{i=1}^{n}(x_i - \mu)^2 = 0$$

$$\frac{n}{2v} = \frac{1}{2v^2}\sum_{i=1}^{n}(x_i - \mu)^2$$

Multiplying by $2v^2$ yields the MLE:

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \mu)^2$$

Expected Value: To find $E[\hat{\sigma}^2]$, we use the linearity of expectation:

$$E[\hat{\sigma}^2] = E\left[\frac{1}{n}\sum_{i=1}^{n}(X_i - \mu)^2\right] = \frac{1}{n}\sum_{i=1}^{n}E[(X_i - \mu)^2]$$

By definition, $E[(X_i - \mu)^2]$ is the population variance σ^2.

$$E[\hat{\sigma}^2] = \frac{1}{n}\sum_{i=1}^{n}\sigma^2 = \frac{1}{n}(n\sigma^2) = \sigma^2$$

This shows the estimator is unbiased.

Problem 4

Determine the MLEs of a and λ for the Pareto density $f(x) = \lambda a^{\lambda} x^{-(\lambda+1)}$ for $x \geq a$.

Solution:
The likelihood function is:

$$L(a, \lambda) = \prod_{i=1}^{n} \lambda a^{\lambda} x_i^{-(\lambda+1)} \cdot I(x_i \geq a) = \lambda^n a^{n\lambda} \left(\prod_{i=1}^{n} x_i \right)^{-(\lambda+1)} \cdot I(\min(x_i) \geq a)$$

Step 1: Estimate a
Look at the term $a^{n\lambda}$. Because $\lambda > 0$ and $n > 0$, this term strictly increases as a increases. To maximize L, we need a to be as large as possible. However, the indicator function bounds a such that $a \leq \min(x_i)$. Therefore, the MLE for a is:

$$\hat{a} = \min(X_1, \ldots, X_n) = X_{(1)}$$

Step 2: Estimate λ
Substitute $\hat{a}$ into the likelihood and take the natural log to find $\ell(\lambda)$:

$$\ell(\lambda) = n \ln(\lambda) + n\lambda \ln(\hat{a}) - (\lambda + 1) \sum_{i=1}^{n} \ln(x_i)$$

Take the derivative with respect to λ and set it to zero:

$$\frac{\partial \ell}{\partial \lambda} = \frac{n}{\lambda} + n \ln(\hat{a}) - \sum_{i=1}^{n} \ln(x_i) = 0$$

$$\frac{n}{\lambda} = \sum_{i=1}^{n} \ln(x_i) - n \ln(\hat{a}) = \sum_{i=1}^{n} (\ln(x_i) - \ln(\hat{a})) = \sum_{i=1}^{n} \ln\left(\frac{x_i}{\hat{a}}\right)$$

Final Answer:

$$\hat{\lambda} = \frac{n}{\sum_{i=1}^{n} \ln(X_i/\hat{a})}$$

Problem 5

Find the MLE of μ_1 and μ_2 from three independent normal samples: $X \sim N(\mu_1, \sigma^2)$, $Y \sim N(\mu_2, \sigma^2)$, and $W \sim N(\mu_1 + \mu_2, \sigma^2)$.

Solution:
Since all variables are independent and share the same variance σ^2, the joint log-likelihood function is proportional to the sum of their squared deviations. To maximize the log-likelihood, we must minimize the following sum of squares, Q:

$$Q(\mu_1, \mu_2) = \sum_{i=1}^{n} (X_i - \mu_1)^2 + \sum_{i=1}^{n} (Y_i - \mu_2)^2 + \sum_{i=1}^{n} (W_i - (\mu_1 + \mu_2))^2$$

We take partial derivatives with respect to μ_1 and μ_2 and set them to zero.
Partial w.r.t μ_1:

$$\frac{\partial Q}{\partial \mu_1} = -2 \sum (X_i - \mu_1) - 2 \sum (W_i - \mu_1 - \mu_2) = 0$$

$$\sum X_i - n\mu_1 + \sum W_i - n\mu_1 - n\mu_2 = 0$$

Divide by n (where $\bar{X}$ and $\bar{W}$ are sample means):

$$\bar{X} + \bar{W} - 2\mu_1 - \mu_2 = 0 \implies 2\mu_1 + \mu_2 = \bar{X} + \bar{W} \quad \text{— (Eq. 1)}$$

Partial w.r.t μ_2:

$$\frac{\partial Q}{\partial \mu_2} = -2\sum(Y_i - \mu_2) - 2\sum(W_i - \mu_1 - \mu_2) = 0$$

$$\sum Y_i - n\mu_2 + \sum W_i - n\mu_1 - n\mu_2 = 0$$

$$\bar{Y} + \bar{W} - \mu_1 - 2\mu_2 = 0 \implies \mu_1 + 2\mu_2 = \bar{Y} + \bar{W} \quad \text{— (Eq. 2)}$$

Now, solve the system of linear equations (Eq 1 & Eq 2):
From Eq. 1: $\mu_2 = \bar{X} + \bar{W} - 2\mu_1$
Substitute into Eq. 2:

$$\mu_1 + 2(\bar{X} + \bar{W} - 2\mu_1) = \bar{Y} + \bar{W}$$

$$-3\mu_1 + 2\bar{X} + 2\bar{W} = \bar{Y} + \bar{W}$$

$$3\mu_1 = 2\bar{X} + \bar{W} - \bar{Y}$$

Final Answer:

$$\hat{\mu}_1 = \frac{2\bar{X} + \bar{W} - \bar{Y}}{3}$$

By symmetry, solving for μ_2 gives:

$$\hat{\mu}_2 = \frac{2\bar{Y} + \bar{W} - \bar{X}}{3}$$

Problem 6

Assuming flood discharges D follow a lognormal distribution, estimate the 100-year flood v where $P\{D \geq v\} = .01$ using the provided table.

Solution:
If D is lognormally distributed, then its natural logarithm follows a normal distribution: $Y = \ln(D) \sim N(\mu, \sigma^2)$.

We are looking for v such that the upper tail is 1%:

$$P(D \geq v) = 0.01 \implies P(\ln(D) \geq \ln(v)) = 0.01$$

Standardizing the normal variable Y:

$$P\left(Z \geq \frac{\ln(v) - \mu}{\sigma}\right) = 0.01$$

The Z-score corresponding to the 99th percentile is approximately $z_{0.01} = 2.326$.

$$\frac{\ln(v) - \mu}{\sigma} = 2.326 \implies \ln(v) = \mu + 2.326\sigma \implies v = \exp(\mu + 2.326\sigma)$$

By the invariance property of MLEs, we can estimate v by plugging in the MLEs for μ and σ derived from the $n = 37$ data points in the table:

1. Transform all 37 data points: $y_i = \ln(d_i)$.
2. Calculate the sample mean of the logs: $\hat{\mu} = \bar{y} = \frac{1}{37}\sum y_i$.
3. Calculate the sample standard deviation (MLE version): $\hat{\sigma} = \sqrt{\frac{1}{37}\sum(y_i - \bar{y})^2}$.
4. Plug into the formula: $\hat{v} = \exp(\hat{\mu} + 2.326\hat{\sigma})$.

(Note: Applying this calculation to the 37 points in the table yields $\hat{\mu} \approx 8.60$ and $\hat{\sigma} \approx 0.53$. Thus $\hat{v} \approx \exp(8.60 + 1.23) \approx \exp(9.83) \approx 18,500$ ft³/s).

Problem 7

X is lognormal with parameters μ and σ^2 (meaning $Y = \ln(X) \sim N(\mu, \sigma^2)$).

(a) Find $E[X]$

Solution:
We need $E[X] = E[e^Y]$. This expression exactly matches the definition of the Moment Generating Function (MGF) of Y, $M_Y(t) = E[e^{tY}]$, evaluated at $t = 1$. Using the MGF of a normal variable: $M_Y(t) = \exp(\mu t + \frac{1}{2}\sigma^2 t^2)$. Substitute $t = 1$:

$$E[X] = M_Y(1) = \exp\left(\mu(1) + \frac{1}{2}\sigma^2(1)^2\right) = \exp\left(\mu + \frac{\sigma^2}{2}\right)$$

(b) Find Var(X)

Solution:
Variance is $\text{Var}(X) = E[X^2] - (E[X])^2$. First, find $E[X^2] = E[(e^Y)^2] = E[e^{2Y}]$. This is the MGF evaluated at $t = 2$:

$$E[X^2] = M_Y(2) = \exp\left(\mu(2) + \frac{1}{2}\sigma^2(2)^2\right) = \exp(2\mu + 2\sigma^2)$$

Now, substitute into the variance formula:

$$\text{Var}(X) = \exp(2\mu + 2\sigma^2) - \left[\exp\left(\mu + \frac{\sigma^2}{2}\right)\right]^2$$

$$\text{Var}(X) = \exp(2\mu + 2\sigma^2) - \exp(2\mu + \sigma^2)$$

Factoring out the common term gives the final expression:

$$\text{Var}(X) = \exp(2\mu + \sigma^2)\left(e^{\sigma^2} - 1\right)$$

(c) Estimate the mean travel time based on data

Solution:
The data is: $42, 28, 53, 57, 67, 39, 35, 50, 44, 39$ $(n = 10)$.
We want to estimate $E[X] = \exp(\mu + \sigma^2/2)$. By the invariance property of MLE, we estimate this by calculating $\hat{\mu}$ and $\hat{\sigma}^2$ from the log-transformed data.

1. **Log-transform the data ($y_i = \ln(x_i)$):**
   $3.738, 3.332, 3.970, 4.043, 4.205, 3.664, 3.555, 3.912, 3.784, 3.664$

2. **Calculate $\hat{\mu}$ (sample mean of y):**

$$\hat{\mu} = \frac{37.867}{10} \approx 3.787$$

3. **Calculate $\hat{\sigma}^2$ (MLE variance of y):**

$$\hat{\sigma}^2 = \frac{1}{10}\sum(y_i - \hat{\mu})^2 \approx 0.0583$$

4. **Estimate Mean Travel Time:**

$$\widehat{E[X]} = \exp\left(3.787 + \frac{0.0583}{2}\right) = \exp(3.787 + 0.02915) = \exp(3.81615)$$

$$\widehat{E[X]} \approx 45.43 \text{ minutes}$$