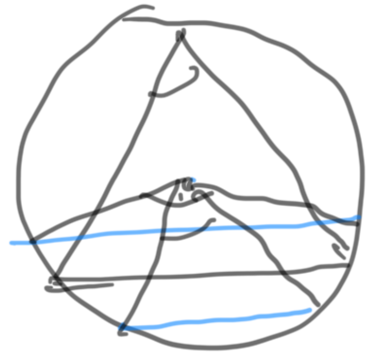
Bertrand Paradox



1st approach

Assume central angle of chord is equally likely.

$$\Omega = [0, 360^\circ)$$

Favorable event

$$E = [120^\circ, 240^\circ]$$

Geometric probability is

$$\mathbb{P}(E) = \frac{\mu(E)}{\mu(\Omega)} = \frac{120^\circ}{360^\circ}$$

$$= \frac{1}{3}$$

2nd approach

We assume that mid point of the chord is equally likely.

Let radius of the circle is unity.

Then, radius of inner circle is $1/2$.

Ω = The circle

E = inner circle

μ — Area.

Geometric probability

$$\mathbb{P}(E) = \frac{\mu(E)}{\mu(\Omega)} = \frac{\pi(1/2)^2}{\pi(1)^2} = \frac{1}{4}$$

Ω — sample space

<u>Algebra/Field</u>: A field $\mathcal{C}$ on Ω is a collection of subsets of Ω, satisfying

(i) $\Omega \in \mathcal{C}$

(ii) if $A \in \mathcal{C}$ then $A^c \in \mathcal{C}$

(iii) if $A, B \in \mathcal{C}$ then $A \cup B \in \mathcal{C}$.

Eg: $\Omega = \{H, T\}$

$\mathcal{C}_1 = \{\phi, \Omega\}$

Yes - all three axioms hold

$\mathcal{C}_2 = \{\phi, \{H\}, \Omega\}$

No, second axiom is not satisfied.

2^Ω - power set of Ω

$\mathcal{Z} =$

$$= \{\phi, \{H\}, \{T\}, \Omega\}$$

<u>Yes</u>.

Let Ω be a sample space and $\mathcal{C}$ be a field on Ω. Then, we have to check if $A, B \in \mathcal{C}$ then $A \cap B \in \mathcal{C}$.

<u>Lemma</u>: Let $\mathcal{C}$ be a field on Ω. If $A, B \in \mathcal{C}$ then

1. $A \cap B \in \mathcal{C}$
2. $A^c \cup B \in \mathcal{C}$

(ii) $A \cup B \in \mathcal{C}$

(iii) $A^c \cap B \in \mathcal{C}$

Proof: $A, B \in \mathcal{C}$

$\Rightarrow A^c, B^c \in \mathcal{C}$

$\Rightarrow A^c \cup B^c \in \mathcal{C}$

$\Rightarrow (A^c \cup B^c)^c \in \mathcal{C}$

$\Rightarrow A \cap B \in \mathcal{C}$

Lemma: Let $\mathcal{C}$ be a field on Ω. If $A_1, \ldots, A_n \in \mathcal{C}$ then $A_1 \cup A_2 \cup \ldots \cup A_n \in \mathcal{C}$.

Hint: Induction

$A_1^c \cap A_2 \cup A_3^c \cap \ldots \cup A_n \in \mathcal{C}$

<u>σ-field</u>: Let $\mathcal{C}$ be a collection of subsets of Ω satisfying following

(i) $\Omega \in \mathcal{C}$

(ii) If $A \in \mathcal{C}$ then $A^c \in \mathcal{C}$

(iii) If a countable collection of subsets $\{A_i\}_{i=1}^{\infty} \subseteq \mathcal{C}$

∞

then $\bigcup_{i=1} A_i \in \mathcal{F}$

Then $\mathcal{F}$ is a σ-field over Ω.

Lemma: A σ-field is a field.

Hint: let $A_3, A_4, \dots = \phi$

Then $\bigcup_{i=1}^{\infty} A_i = A_1 \cup A_2$

Lemma: A field may not be a σ-field.

$\rightarrow$ Tutorial

Probability measure:

$(\Omega, \mathcal{F})$ — measurable space

$\mathcal{F}$: σ-field on Ω

Def: A probability measure $\mathbb{P}$ on a measurable space $(\Omega, \mathcal{F})$ is defined by following axioms.

(I) $\mathbb{P}(\Omega) = 1$

(II) $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$

(III) If $\{A_i\}_{i=1}^{\infty} \subseteq \mathcal{F}$ and A_i's are pairwise disjoint i.e. $A_i \cap A_j = \phi \ \forall i \neq j$,

then
$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$$

Eg: Tossing of a coin

$\Omega = \{H, T\}$

$\mathcal{C} = 2^{\Omega}$, power set of Ω

$\mathcal{C} = \{\phi, \{H\}, \{T\}, \Omega\}$

		ϕ	$\{H\}$	$\{T\}$	Ω
$\mathbb{P}_1$	:	0	$\frac{1}{2}$	$\frac{1}{2}$	1
$\mathbb{P}_2$	:	0	$\frac{1}{3}$	$\frac{2}{3}$	1

Remark: A probability measure on $(\Omega, \mathcal{C})$ is a set-function

$$\mathbb{P} : \mathcal{C} \longrightarrow [0,1]$$