March 13

Let there is a coin with prob. of getting head in single toss be $p \in (0,1)$.

Our goal is to "approximate" p.

We toss the coin n-times independently in identical condition.

Let … X … be the

$X_1, \ldots, X_n$ are the

RVs associated with those tosses, where

$$X_i = \begin{cases} 1 & \text{if } i^{th} \text{ outcome is head} \\ 0 & \text{otherwise} \end{cases}$$

An approximation of p from here is the sample mean

$$\bar{X}_n = \frac{X_1 + \cdots + X_n}{n}$$

① $$E(\bar{X}_n) = E\left(\frac{X_1 + \cdots + X_n}{n}\right)$$

$$= \frac{1}{n}\left[E(X_1) + \cdots + E(X_n)\right]$$

$$= \frac{1}{n} \, n \, E(X_1) = E(X_1)$$

$$= p$$

(ii) We know sum of IID Bernoulli RVs is a Binomial RV.

$$X_1 + X_2 + \dots + X_n =: Y_n$$

Then

$$Y_n \sim \text{Binomial}(n, p)$$

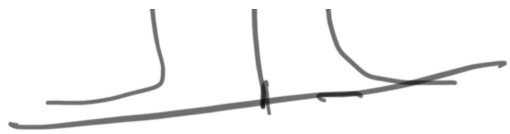
Observe that

$$Y_n = n \bar{X}_n$$

$$\Rightarrow \bar{X}_n = \frac{1}{n} Y_n$$

where $Y_n \sim \text{Binomial}(n, p)$

$$\begin{aligned}
Var(\bar{X}_n) &= Var\left(\frac{1}{n} Y_n\right) \\
&= \frac{1}{n^2} Var(Y_n) \\
&= \frac{1}{n^2} np(1-p) \\
&= \frac{p(1-p)}{n}.
\end{aligned}$$



$N(0, 0.1)$

less variance

more concentrated

$N(0, 2)$

high variance

scattered.

Eg: let $X_1, \ldots, X_n$ are IID observations from

$N(\mu, \sigma^2)$

μ unknown

σ^2 known

An approximation of μ

$\underline{\mu}$ $X_1 + \ldots + X_n$

$$\dot{X}_n = \frac{\quad}{n}$$

(i) $E(\bar{X}_n) = \mu$

(ii) Using MGF,

$$\bar{X}_n \sim N(\mu, \sigma^2/n)$$

Next, when variance is unknown.

Eg: Let $X_1, \ldots, X_n$ are IID RVs with mean μ and variance σ^2.

Define

$$\bar{X}_n = \frac{1}{n}(X_1 + \cdots + X_n)$$

$$S_n^2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2 - (\bar{X}_n)^2$$

(1) $\mathbb{E}(S_n^2)$

$$= \frac{1}{n}\sum_{i=1}^{n} \underline{\mathbb{E}(X_i^2)} - \underline{\mathbb{E}\left[(\bar{X}_n)^2\right]}$$

Observe that

$$\mathrm{Var}(X_1) = \mathbb{E}(X_1^2) - [\underline{\mathbb{E}(X_1)}]^2$$

$$\sigma^2 = \mathbb{E}(X_1^2) - \mu^2$$

$$\Rightarrow \sigma^2 = \ldots$$

$$\Rightarrow \boxed{\mathbb{E}(x_1^2) = \sigma^2 + \mu^2}$$

Next

$$\operatorname{var}(\bar{X}_n) = \mathbb{E}\left[(\bar{X}_n)^2\right] - \left[\underbrace{\mathbb{E}(\bar{X}_n)}\right]^2$$

$$\Rightarrow \frac{\sigma^2}{n} = \mathbb{E}\left[(\bar{X}_n)^2\right] - \mu^2$$

$$\Rightarrow \boxed{\mathbb{E}\left[(\bar{X}_n)^2\right] = \frac{\sigma^2}{n} + \mu^2}$$

Therefore,

$$\mathbb{E}(S_n^2) = \frac{1}{n}\sum_{i=1}^{n}(\sigma^2 + \mu^2)$$

$$[\sigma^2 + \mu^2)$$

$$= \left[\frac{\sigma^2}{n} + \mu^2\right]$$

$$= \sigma^2 + \mu^2 - \frac{\sigma^2}{n} - \mu^2$$

$$= \sigma^2\left(1 - \frac{1}{n}\right)$$

$$\Rightarrow \mathbb{E}(S_n^2) = \sigma^2\left(1 - \frac{1}{n}\right)$$

$$= \frac{n-1}{n}\sigma^2$$

for large n

$\frac{n-1}{n}$ is close to 1.

Claim:

$$S^2 = \frac{1}{\quad}\sum^{n}(x_i - \bar{X}_n)^2$$

Let $S_{n-1}^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X}_n)^2$

then $E(S_{n-1}^2) = \sigma^2$

proof: $S_{n-1}^2 = \frac{n}{n-1} \cdot \underbrace{\frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X}_n)^2}$

$$= \frac{n}{n-1} \cdot S_n^2$$

$$\Rightarrow E(S_{n-1}^2) = \frac{n}{n-1} E(S_n^2)$$

$$= \frac{n}{n-1} \cdot \frac{n-1}{n} \sigma^2$$

$$= \sigma^2$$