MTL108
Preliminaries

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Set Theory

We begin with a definition of a set and the basic notation we use to represent sets.

Definition 1. A *set* X is a collection of well-defined elements from a known universe.

The notation $a \in X$ denotes that a is a member of a set X. We use the notation $a \notin X$ to denote that a is not a member of a set X.

Some standard and well-known sets:

- $\mathbb{N}$: natural numbers
- $\mathbb{Z}$: integers
- $\mathbb{Q}$: rational numbers
- $\mathbb{R}$: real numbers

In general, sets are denoted inside of curly braces $\{\}$. Let's look at some examples of defining sets. First, if possible, we could simply list all the elements in a set. Consider:

$$X = \{2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22\}.$$

This tells us that X is the set whose elements are all the even numbers from 2 up to 22 (inclusive). As noted, sets are unordered objects, so we could equally well have written

$$X = \{2, 10, 18, 4, 12, 20, 6, 14, 22, 8, 16\},$$

and the set X would be no different. In addition, each element of the universe is either in a set or not; its inclusion is binary. Hence, listing elements more than once also does not change the set. So, for example, we could write

$$X = \{2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22\},$$

and the set X still is no different.

Two definitions:

Definition 2. The *universe or universal set* is a set containing all elements in the context; that is, the set contains everything. It is usually denoted by Ω.

Definition 3. The *empty set* is a set containing no elements. It is usually denoted by $\emptyset$.

Now, suppose we wished to define a set Y containing all the even numbers from 2 up to 5274. Obviously we would not like to list all these numbers! So we need a different way to describe Y. One obvious approach is to write Y as

$$Y = \{2, 4, 6, 8, \ldots, 5274\}.$$

In general, this is an acceptable description of Y, sometimes called an *implied list.* Also in general, I would warn you to be careful with implied lists, as context always matters.

For example, in the above implied list, you expect your reader to make the assumption that the ... means to consider all even numbers up to 5274. However, perhaps what you really intended to list is "all even numbers that are either powers of 2 or divisible by 3." Or maybe what you are listing is really "integers between 2 and 5274 that are not odd primes." Certainly every number that is written in this presentation of Y qualifies under either of these definitions, so it is of critical import that your specific intention is clear from context here.

To avoid this kind of potential pitfall, we have yet another notation that can be useful to describe sets, called *set-builder notation.* For set-builder notation, we describe a set in two parts: first, the universe from which the numbers come, and second, the rules for belonging to the set. For example, if we wanted to clarify that Y is all the positive even numbers up to 5274, we could write

$$Y = \{x \in \mathbb{N} \mid x \text{ is even and } x \leq 5274\}.$$

Here, we see that the first part of the notation, $x \in \mathbb{N}$, describes where our numbers come from (that is, the positive integers). The second part of our notation describes the rules for being a member of Y: any member of Y must be even and no larger than 5274. The central bar in this notation is sometimes written instead as a :, and is usually read as "such that," so that speaking this presentation, I would say "Y is the set of natural numbers x such that x is even and x is at most 5274."

Using set-builder notation, notice that the description to the right of the central bar is in fact a proposition about x, that given a member of the range $\mathbb{N}$ can be true or false. In general, this is how set-builder notation works. Given a proposition $p(x)$, where x is a variable whose range is Ω, we write

$$X = \{x \in \Omega \mid p(x)\}$$

to denote the set of elements in the universe Ω for which $p(x)$ is true. Using this type of notation, we have no ambiguity about what elements are in a set, since for each $x \in \Omega$ we clearly either have $p(x)$ true or false.

Now, it is easy to see that:

- $\mathbb{N} = \{1, 2, 3, \ldots\}$
- $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$
- $\mathbb{Q} = \left\{x \in \mathbb{R} \mid x = \dfrac{p}{q} \text{ where } p, q \in \mathbb{Z} \text{ and } q \neq 0\right\}$

1 Subset and superset

Definition 4. Let A and B be sets in universe Ω. We say A is a *subset* of B if $x \in A$ then $x \in B$. We write $A \subseteq B$ to denote that A is a subset of B. If A is a subset of B, we call B a *superset* of A, and write $B \supseteq A$ to denote that B is a superset of A.

Let's take Y, as above, to be the set

$$Y = \{x \in \mathbb{N} \mid x \text{ is even and } x \leq 5274\}.$$

Now, suppose that we wanted to isolate only those members of Y that are divisible by 3. We could create a new set Z, explicitly to be a subset of Y, as

$$Z = \{x \in Y \mid x \text{ is divisible by } 3\}.$$

Here, we have used the set Y as the range to consider when constructing the set Z, so only members of Y can be elements of Z. By definition, we thus have $Z \subseteq Y$.

Let's consider an example of showing one set is a subset of another, following this definition.

Example 1. Let $X = \{x \in \mathbb{Z} \mid x \text{ is even}\}$ and let $Y = \{x \in \mathbb{Z} \mid x = 4k + 2 \text{ for some } k \in \mathbb{Z}\}$. Then $Y \subseteq X$.

Proof. Per the definition of subset, we wish to show that $x \in Y \Rightarrow x \in X$. We work by direct proof.
Suppose $x \in Y$, so that there exists $k \in \mathbb{Z}$ such that $x = 4k + 2$. Then $x = 2(2k + 2)$, and hence x is even. By definition, then, $x \in X$. Therefore, $x \in Y \Rightarrow x \in X$, and thus $Y \subseteq X$. □

We note, moreover, that by definition, $\emptyset \subseteq X$ and $X \subseteq \Omega$ for every set X.

We note that the symbol $\subset$ is sometimes used in place of $\subseteq$ to indicate that equality is impossible. As with $<$ and $\leq$, the difference between the two symbols is that in the latter case we allow the two things being compared to be the same, and in the former we force that they are different.

To dig into this a little further, let's consider what it means for two sets to be equal, and how we could prove they are equal if in fact they are. We start with a perhaps trivial definition, which we can then use to think about proof techniques for showing set equalities.

Definition 5. Let A and B be sets in universe Ω. We say that $A = B$ if $A \subseteq B$ and $B \subseteq A$.

Example 2. Let

$$A = \{x \in \mathbb{Z} \mid x = 4k + 3 \text{ for some } k \in \mathbb{Z}\},$$

and let

$$B = \{x \in \mathbb{Z} \mid x = 4k - 1 \text{ for some } k \in \mathbb{Z}\}.$$

Then $A = B$.

Proof. We show double containment, as described above.
First, to show that $A \subseteq B$, let $x \in A$. Then there exists $k \in \mathbb{Z}$ such that $x = 4k + 3$. But then $x = 4k + 3 = 4(k + 1) - 1$, and hence by definition $x \in B$. Therefore, $A \subseteq B$.
For the other containment, let $x \in B$. Then there exists $k \in \mathbb{Z}$ such that $x = 4k - 1$. But then $x = 4k - 1 = 4(k - 1) + 3$, and hence by definition $x \in A$. Therefore, $B \subseteq A$. □

Finally, we note that in some cases, thinking about what the possible subsets of a given set might look like can be interesting. We define this as follows:

Definition 6. Let X be a set. The *power set* of X, denoted by $\mathcal{P}(X)$ or 2^X is the set whose elements are all the possible subsets of X. That is to say, $\mathcal{P}(X) = \{A \mid A \subseteq X\}$.

For example, if

$$X = \{1, 2, 3, 4\},$$

then we have

$$\mathcal{P}(X) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}, \{1,2,3\}, \\ \{1,2,4\}, \{1,3,4\}, \{2,3,4\}, \{1,2,3,4\}\}.$$

That is to say, $\mathcal{P}(X)$ is the set of all possible sets we can make out of the elements of X. Notice that, as we discussed above, since $\emptyset \subseteq X$, we must have $\emptyset \in \mathcal{P}(X)$, and likewise $X \in \mathcal{P}(X)$.

2 Set operations

Now that we have a sense of what sets look like, and how to think about subsets, let's dive into the kinds of operations we can do on sets.

Definition 7. Let A, B be sets. Define the *intersection* of A and B, denoted by $A \cap B$, to be the set

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}.$$

If we think of A and B using set builder notation, as follows:

$$A = \{x \mid p(x)\}, \quad B = \{x \mid q(x)\}, \tag{1}$$

then we have

$$A \cap B = \{x \mid p(x) \text{ and } q(x)\}.$$

Likewise,

Definition 8. Let A, B be sets. Define the *union* of A and B, denoted by $A \cup B$, to be the set

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}.$$

Indeed, if A and B are as in (1), then we have

$$A \cup B = \{x \mid p(x) \text{ or } q(x)\}.$$

To ensure that $A \cap B$ and $A \cup B$ are understood, see the Venn diagram and caption in Figure 1. We shall occasionally use Venn diagrams to convey an understanding of sets and their relationships.

We easily obtain the following theorem:

Theorem 1. *Let A, B, C be sets. Then*

- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$, *and*
- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Next, we define set operations known as complementation.

Definition 9. Let A be a set in the universe Ω. The *complement* of A is denoted by $\Omega \setminus A$ or A^c and defined by $A^c = \{x \in \Omega \mid x \notin A\}$.

Indeed, if we allow the underlying universe to vary, we end up with a different understanding of complement, known as the relative complement.

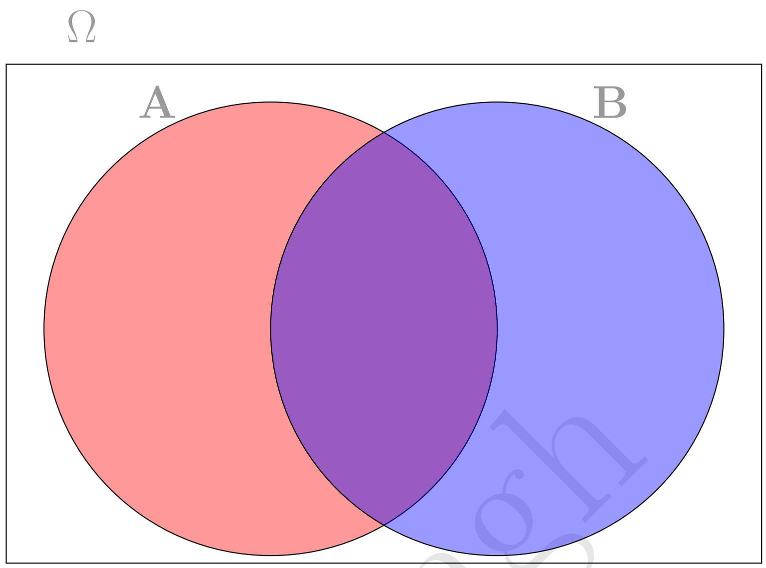


Figure 1: Imagine that the rectangle describes the possible universe Ω. If A is the red set, and B is the blue set, then the purplish set where the two overlap is $A \cap B$. The set $A \cup B$ is all colored portions of the diagram.

Definition 10. Let A, B be sets. The *relative complement* of B in A, denoted by $A \backslash B$ is the set defined by

$$A \backslash B = \{x \mid x \in A \text{ and } x \notin B\}.$$

That is to say, the relative complement of B in A is the set of elements from A that do not appear in B. This is sometimes read as "set minus," that is, $A \backslash B$ is read as "A set-minus B." A warning here: there is no rule that B must be contained in A to "subtract" B from A. We just take out whatever elements of B *are* there, and ignore the rest. So $A \backslash B = A \backslash (A \cap B)$, since we only concern ourselves with those members of B that are also members of A.

Since the operations of union, intersection, and complementation for sets have obvious connections to the operations of conjunction, disjunction, and negation for propositions, we immediately obtain a version of De Morgan's Laws for sets.

Theorem 2 (De Morgan's Laws for Sets)**.** *Let A, B be sets in the universe Ω. Then*

- $(A \cup B)^c = A^c \cap B^c$, *and*
- $(A \cap B)^c = A^c \cup B^c$.

Proof. To prove $(A \cup B)^c = A^c \cap B^c$, we need to show that $(A \cup B)^c \subseteq A^c \cap B^c$ and $A^c \cap B^c \subseteq (A \cup B)^c$.

Part 1: $(A \cup B)^c \subseteq A^c \cap B^c$ Suppose $x \in (A \cup B)^c$. This means $x \notin A \cup B$. By the definition of union, $x \notin A$ and $x \notin B$. Thus, $x \in A^c$ (since $x \notin A$) and $x \in B^c$ (since $x \notin B$). Therefore, $x \in A^c \cap B^c$. Hence, $(A \cup B)^c \subseteq A^c \cap B^c$.

Part 2: $A^c \cap B^c \subseteq (A \cup B)^c$ Suppose $x \in A^c \cap B^c$. This means $x \in A^c$ and $x \in B^c$, so $x \notin A$ and $x \notin B$. Since x is not in A or B, $x \notin A \cup B$, implying $x \in (A \cup B)^c$. Thus, $A^c \cap B^c \subseteq (A \cup B)^c$. Since both inclusions hold, we conclude $(A \cup B)^c = A^c \cap B^c$.

Next, to prove $(A \cap B)^c = A^c \cup B^c$, we show that $(A \cap B)^c \subseteq A^c \cup B^c$ and $A^c \cup B^c \subseteq (A \cap B)^c$.

Part 1: $(A\cap B)^c \subseteq A^c \cup B^c$ Suppose $x \in (A\cap B)^c$. This means $x \notin A \cap B$. By the definition of intersection, $x \notin A\cap B$ implies $x \notin A$ or $x \notin B$. Thus, $x \in A^c$ or $x \in B^c$, so $x \in A^c \cup B^c$. Therefore, $(A\cap B)^c \subseteq A^c \cup B^c$.

Part 2: $A^c \cup B^c \subseteq (A\cap B)^c$ Suppose $x \in A^c \cup B^c$. This means $x \in A^c$ or $x \in B^c$, so $x \notin A$ or $x \notin B$. If $x \notin A$ or $x \notin B$, then x cannot be in both A and B, so $x \notin A \cap B$. Thus, $x \in (A\cap B)^c$. Hence, $A^c \cup B^c \subseteq (A\cap B)^c$. Since both inclusions hold, we conclude $(A\cap B)^c = A^c \cup B^c$. □

Now, it is often the case that we wish to intersect or union more than just one set. To do so, we recursively define the following notation:

Given sets $A_1, A_2, \ldots, A_n$ in a universe Ω, define

$$\bigcup_{i=m}^{k} A_i = \emptyset \text{ if } k < m; \quad \bigcup_{i=m}^{k} A_i = \left(\bigcup_{i=m}^{k-1} A_i\right) \cup A_k \text{ if } k \geq m, \text{ and}$$

$$\bigcap_{i=m}^{k} A_i = \Omega \text{ if } k < m; \quad \bigcap_{i=m}^{k} A_i = \left(\bigcap_{i=m}^{k-1} A_i\right) \cap A_k \text{ if } k \geq m.$$

This notation is similar to the recursive notation we defined for summations and products in the Induction notes. We note that under this definition, we can show the following:

Proposition 1. Let $A_1, A_2, \ldots, A_n$ be sets in a universe Ω. Then we have

- $\displaystyle\bigcup_{i=1}^{n} A_i = \{x \in \Omega \mid \exists i \text{ with } 1 \leq i \leq n,\ x \in A_i\}$, and
- $\displaystyle\bigcap_{i=1}^{n} A_i = \{x \in \Omega \mid \forall i \text{ with } 1 \leq i \leq n,\ x \in A_i\}$.

Here, we will prove the first statement, and leave the second as an exercise.

Proof. Let $A_1, A_2, \ldots, A_n$ be sets in Ω. We prove that $\displaystyle\bigcup_{i=1}^{n} A_i = \{x \in \Omega \mid \exists i \text{ with } 1 \leq i \leq n,\ x \in A_i\}$ by induction on n.

For the base case, when $n = 1$, we have that $\displaystyle\bigcup_{i=1}^{n} A_i = A_1$. On the other hand, $\{x \in \Omega \mid \exists i \text{ with } 1 \leq i \leq 1,\ x \in A_i\} = A_1$, since the only value i can take is 1. Hence, the result holds in the case that $n = 1$.

Now, let us suppose that for some $k \geq 1$, it is true that $\displaystyle\bigcup_{i=1}^{k} A_i = \{x \in \Omega \mid \exists i \text{ with } 1 \leq i \leq k,\ x \in A_i\}$. Consider the case of $k+1$. We have

$$\begin{aligned}
\bigcup_{i=1}^{k+1} A_i &= \bigcup_{i=1}^{k} A_i \cup A_{k+1} \quad \text{(by definition)}\\
&= \{x \in \Omega \mid \exists i \text{ with } 1 \leq i \leq k,\ x \in A_i\} \cup A_{k+1} \quad \text{(by the inductive hypothesis)}\\
&= \{x \in \Omega \mid \exists i \text{ with } 1 \leq i \leq k,\ x \in A_i \text{ or } x \in A_{k+1}\} \quad \text{(by definition of union)}\\
&= \{x \in \Omega \mid \exists i \text{ with } 1 \leq i \leq k+1,\ x \in A_i\} \quad \text{(since each } x \text{ is in one of } A_1, \ldots, A_k \text{ or in } A_{k+1}).
\end{aligned}$$

Hence, the result also holds for $k+1$.

By induction, then, for any choice of n, we have that $\bigcup_{i=1}^{n} A_i = \{x \in \Omega \mid \exists i \text{ with } 1 \leq i \leq n,\ x \in A_i\}$.

□

We can use this more general definition of a multiway union/intersection to develop a more sophisticated set of De Morgan's Laws for sets. The proof of this theorem is a homework exercise, but as with the first version of De Morgan's Laws, it can proven in multiple ways. Induction is an option, as is using Proposition 1 and showing double containment to demonstrate set equality.

Theorem 3 (De Morgan's Laws for Sets, v. 2)**.** *Let* $A_1, A_2, \ldots, A_n$ *be sets in the universe* Ω*. Then*

- $\left(\bigcup_{i=1}^{n} A_i\right)^c = \bigcap_{i=1}^{n} A_i^c$, *and*
- $\left(\bigcap_{i=1}^{n} A_i\right)^c = \bigcup_{i=1}^{n} A_i^c$.

Our final operation on sets to define here is the Cartesian product. This operation is a little different, as the output of a Cartesian product does not live in the same universe as the original sets.

Definition 11. Let A, B be sets, from possibly different universes Ω_1 and Ω_2. Define the *Cartesian product* of A and B, denoted by $A \times B$, as the set

$$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}.$$

That is to say, the Cartesian product consists of all ordered pairs of elements, in which the first element comes from A and the second element comes from B.

3 Cardinality

Definition 12. The *cardinality* of a set is a measure of the number of elements in the set. It is denoted by $|A|$ for a set A.

For finite sets, the cardinality is simply the count of elements. For infinite sets, cardinality is determined by the existence of a bijection (one-to-one correspondence) with another set, such as the natural numbers.

Example 3. For the set $A = \{1, 2, 3\}$, the cardinality is $|A| = 3$.

Definition 13. A set is *finite* if it contains a specific, countable number of elements (including possibly zero elements).

The cardinality of a finite set is a non-negative integer.

Example 4. The set $B = \{a, b, c, d\}$ is finite with cardinality $|B| = 4$. The empty set $\emptyset$ is also finite with cardinality $|\emptyset| = 0$.

Definition 14. A set is *infinite* if it is not finite, meaning it has an unbounded number of elements, and no finite number can represent its cardinality.

Example 5. The set of natural numbers $\mathbb{N} = \{1, 2, 3, \dots\}$ is infinite because it has no upper bound on the number of elements.

Definition 15. A set is *countable* if its elements can be put into a one-to-one correspondence with the natural numbers $\mathbb{N}$ (i.e., it is either finite or has the same cardinality as $\mathbb{N}$). Countably infinite sets have cardinality denoted by $\aleph_0$ (aleph-null).

Example 6. The set of integers $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ is countably infinite because it can be listed as $\{0, 1, -1, 2, -2, \dots\}$, forming a bijection with $\mathbb{N}$. Thus, $|\mathbb{Z}| = \aleph_0$.

Definition 16. A set is *uncountable* if it is infinite and its elements cannot be put into a one-to-one correspondence with the natural numbers. Its cardinality is strictly greater than $\aleph_0$.

Example 7. The set of real numbers $\mathbb{R}$ is uncountable. Cantor's diagonal argument shows that no bijection exists between $\mathbb{N}$ and $\mathbb{R}$, and its cardinality is denoted $2^{\aleph_0}$, also known as the continuum.

Theorem 4. *The set of even numbers $E = \{2, 4, 6, \dots\}$ is countable.*

Proof. To show that E is countable, we define a function $f : \mathbb{N} \to E$ by $f(n) = 2n$. We prove that f is a bijection.

First, we check injectivity. Suppose $f(n) = f(m)$. Then:

$$2n = 2m \implies n = m.$$

Thus, f is injective.

Next, we check surjectivity. For any even number $k \in E$, there exists $n \in \mathbb{N}$ such that $f(n) = k$. Since k is even, we can write $k = 2n$, so:

$$n = \frac{k}{2},$$

which is a natural number because k is even. Thus, $f(n) = 2n = k$, and f is surjective.

Since f is both injective and surjective, it is a bijection. Therefore, $|E| = |\mathbb{N}|$, and the set of even numbers is countable. $\square$

Theorem 5. *The set of odd numbers $O = \{1, 3, 5, \dots\}$ is countable.*

Proof. To show that O is countable, we define a function $g : \mathbb{N} \to O$ by $g(n) = 2n - 1$. We prove that g is a bijection.

First, we check injectivity. Suppose $g(n) = g(m)$. Then:

$$2n - 1 = 2m - 1 \implies 2n = 2m \implies n = m.$$

Thus, g is injective.

Next, we check surjectivity. For any odd number $k \in O$, we need $n \in \mathbb{N}$ such that $g(n) = k$. Set:

$$g(n) = 2n - 1 = k \implies 2n = k + 1 \implies n = \frac{k+1}{2}.$$

Since k is odd, $k + 1$ is even, so $\frac{k+1}{2}$ is a natural number. For example, if $k = 3$, then:

$$n = \frac{3+1}{2} = 2, \quad \text{and} \quad g(2) = 2 \cdot 2 - 1 = 3.$$

Thus, g is surjective.

Since g is both injective and surjective, it is a bijection. Therefore, $|O| = |\mathbb{N}|$, and the set of odd numbers is countable. $\square$

Some counting rules

Theorem 6. *Consider a compound experiment consisting of two sub-experiments, Experiment A and Experiment B. Suppose that Experiment A has a possible outcomes, and for each of those outcomes Experiment B has b possible outcomes. Then the compound experiment has ab possible outcomes.*

Example 8. A Pizza Shop offers pizzas with 4 different types of crust and a choice of 15 toppings. How many different one-topping pizzas can be made at the shop?

Theorem 7 (Sampling with replacement)**.** *Consider n objects and making k choices from them, one at a time with replacement (i.e., choosing a certain object does not preclude it from being chosen again). Then there are n^k possible outcomes.*

Example 9. How many license plates can you make out of three letters followed by three numerical digits?

Theorem 8 (Sampling without replacement)**.** *Consider n objects and making k choices from them, one at a time without replacement (i.e., choosing a certain object precludes it from being chosen again). Then there are $n(n-1)\dots(n-k+1)$ possible outcomes, for $k \leq n$ (and 0 possibilities for $k > n$).*

Example 10. How many license plates can you make out of three different letters followed by three different numerical digits?

Example 11. (Permutations and factorials) A permutation of $1, 2, \dots, n$ is an arrangement of them in some order, e.g., $3, 5, 1, 2, 4$ is a permutation of $1, 2, 3, 4, 5$, i.e., order matters. For example, there are $n!$ ways in which n people can line up for ice cream. (Recall that $n! = n(n-1)(n-2)\dots 1$ for any positive integer n, and $0! = 1$.)

Example 12. Consider a group of four people.
(a) How many ways are there to choose a two-person committee?
(b) How many ways are there to break the people into two teams of two?

Definition 17 (Binomial Coefficient)**.** For any nonnegative integers k and n, the binomial coefficient $\binom{n}{k}$, read as "n choose k", is the number of subsets of size k for a set of size n. Precisely,

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Example 13. In a club with n people, there are $n(n-1)(n-2)$ ways to choose a president, vice president, and treasurer, and there are $\binom{n}{3}$ ways to choose 3 positions without predetermined titles.

Choosing vs. Arranging in Counting

In combinatorics, a crucial distinction is between *choosing* (combinations) and *arranging* (permutations).

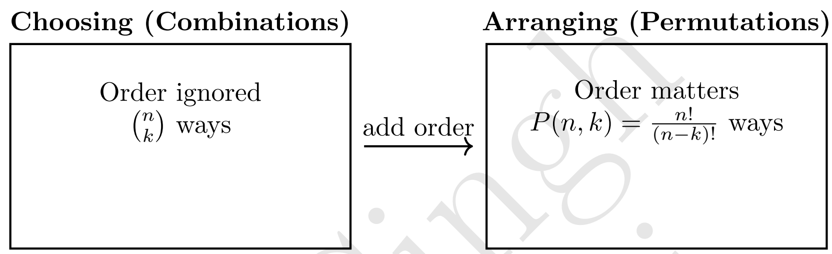
Definition 18 (Choosing)**.** Choosing k objects from n without regard to order is called a *combination*. The number of ways is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Definition 19 (Arranging)**.** Arranging k objects selected from n in a specific order is called a *permutation*. The number of ways is

$$P(n,k) = \frac{n!}{(n-k)!}.$$

The key difference is that permutations treat order as important, while combinations do not.



Example 14. (choosing): Selecting 3 students from a class of 10 to form a committee. The set $\{A, B, C\}$ is the same as $\{C, B, A\}$.

$$\binom{10}{3} = 120.$$

Example 15. (arranging): Selecting 3 students from 10 to be president, vice-president, and secretary. Now $\{A, B, C\}$ arranged as (A, B, C) differs from (C, B, A).

$$P(10, 3) = 720.$$

Pascal's Triangle and the Binomial Theorem

Pascal's Triangle. The binomial coefficients $\binom{n}{k}$ can be arranged in Pascal's Triangle. Each entry is the sum of the two directly above it:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}, \qquad 1 \le k \le n-1.$$

```
          1
        1   1
      1   2   1
    1   3   3   1
  1   4   6   4   1
1   5  10  10   5   1
```

Binomial Theorem. For any integer $n \ge 0$,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.$$

Example. Expand $(x+y)^4$:

$$(x+y)^4 = \binom{4}{0}x^4 + \binom{4}{1}x^3y + \binom{4}{2}x^2y^2 + \binom{4}{3}xy^3 + \binom{4}{4}y^4.$$

$$(x+y)^4 = 1\cdot x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1\cdot y^4.$$

Set theory related results:

Theorem 9. *For any two sets A and B and universal set Ω. We have*

1. $0 \le |A| \le |\Omega|$.
2. $|A\cup B| = |A| + |B| - |A\cap B|$.
3. $|A| + |A^c| = |\Omega|$, $A^c = \Omega \setminus A$.
4. *If* $A \subseteq B$ *then* $|A| \le |B|$.

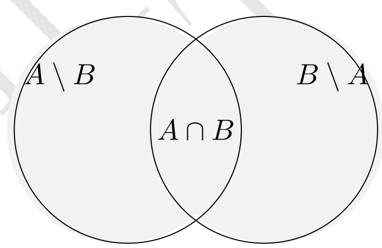
Proof. Throughout we use the notation $|S|$ for the (finite) number of elements of a set S.

(1) Nonnegativity and upper bound. By definition, the cardinality $|A|$ is the number of elements of A. The smallest possible number of elements is 0 (attained by the empty set $\varnothing$), so $|A| \ge 0$. Since every element of A is also an element of Ω, A cannot have more elements than Ω, hence $|A| \le |\Omega|$. Thus $0 \le |A| \le |\Omega|$.

(2) Inclusion–exclusion formula. Decompose the union $A\cup B$ into three disjoint regions:

$$A\setminus B, \qquad A\cap B, \qquad B\setminus A.$$

Venn diagram and labeled regions.



$A\cup B$ partitioned into three disjoint regions

These three sets are pairwise disjoint and their union equals $A\cup B$. Let

$$a := |A\setminus B|, \qquad c := |A\cap B|, \qquad b := |B\setminus A|.$$

Then

$$|A| = |A\setminus B| + |A\cap B| = a + c,$$
$$|B| = |B\setminus A| + |A\cap B| = b + c,$$

and

$$|A\cup B| = |A\setminus B| + |A\cap B| + |B\setminus A| = a + c + b.$$

Adding the two expressions for $|A|$ and $|B|$ gives

$$|A| + |B| = (a + c) + (b + c) = a + b + 2c.$$

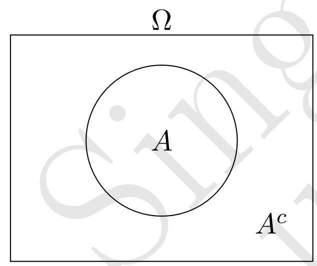
Comparing with $|A \cup B| = a + b + c$ we obtain

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

(3) Complementary partition. By definition $A^c = \Omega \setminus A$ is the set of all elements of Ω that are not in A. The sets A and A^c are disjoint and their union is the whole universe:

$$A \cap A^c = \varnothing, \qquad A \cup A^c = \Omega.$$

Diagram for complement.



Disjoint union: $\Omega = A \cup A^c$

Hence

$$|\Omega| = |A \cup A^c| = |A| + |A^c|$$

because the union is a disjoint union. This proves $|A| + |A^c| = |\Omega|$.

(4) Monotonicity: $A \subseteq B \implies |A| \le |B|$. Assume $A \subseteq B$. Then we can write B as the disjoint union

$$B = A \cup (B \setminus A),$$

where A and $B \setminus A$ are disjoint. Taking cardinalities and using additivity over disjoint unions,

$$|B| = |A| + |B \setminus A|.$$

Since $|B \setminus A| \ge 0$, it follows that $|B| \ge |A|$, as required. $\square$