Feb 4

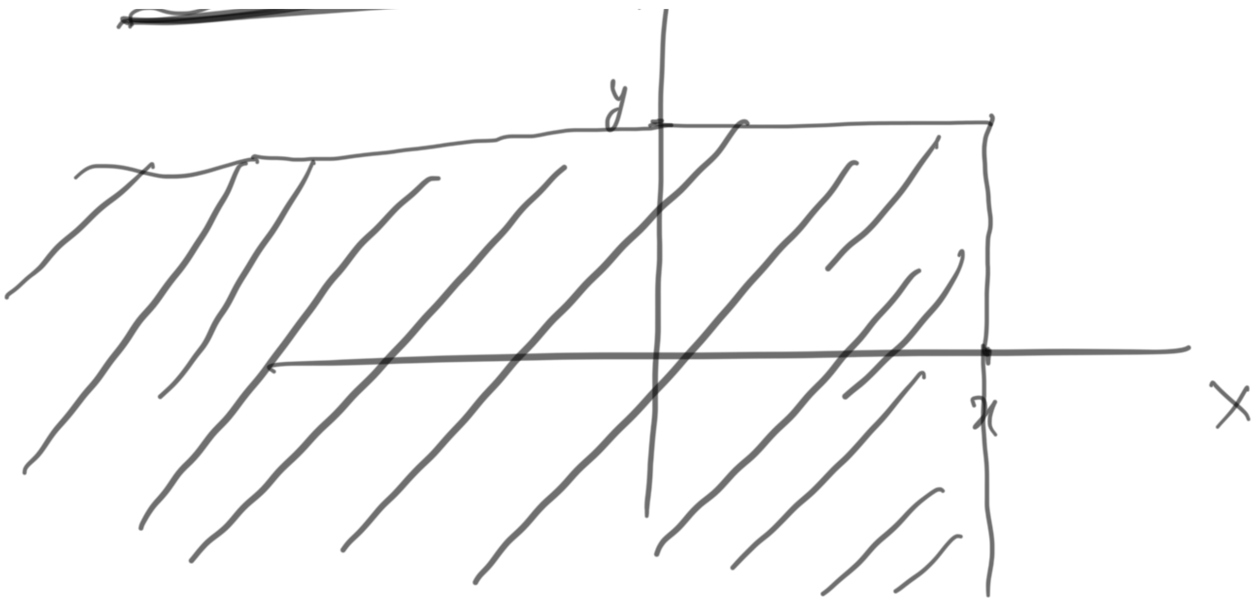
Bivariate Random variable (vector)

~~Def~~: Let $(X, Y) : (\Omega, \mathcal{F}) \to (\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2))$ and $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Then the joint variable (X, Y) is said to be a bivariate RV.

Characterization: For a bivariate RV (X, Y) the CDF is defined as

$$F_{XY}(x, y) = \mathbb{P}(X \leq x, Y \leq y)$$

Y,



Eg: Consider tossing of three fair coins

$$\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

$$\mathcal{C} = 2^{\Omega}$$

$\mathbb{P} \longrightarrow$ equally likely

X = number of heads

Y = number of tails

$$(X, Y) \in \{(0,3), (1,2), (2,1), (3,0)\}$$

$\ldots$

$HHH \longrightarrow (3, 0)$

$\ldots$

$$\underbrace{\mathbb{P}(X=1, Y=2)}$$

$$= \mathbb{P}(\{HTT, THT, TTH\})$$

$$= \frac{3}{8}$$

Joint PMF (for discrete)

For a discrete bivariate RV support set is countable. The joint PMF is defined as

$\mathbb{P}(\ldots)$

$$p_{XY}(x,y) = \mathbb{P}(X = x, Y = y)$$

Marginal PMF (discrete)

Marginal PMF of X is defined as

$$p_X(x) = \sum_{y \in \text{Support}} p_{XY}(x,y)$$

Eg: The joint PMF is

$$p_{XY}(x,y) = \begin{cases} \frac{1}{8} & \text{if } (x,y) \in \{(0,3), (3,0)\} \\ & \quad \in \{(1,2), (2,1)\} \end{cases}$$

$\Big\lfloor \frac{2}{8}$ if $(x,y) \in \{ \ldots \}$

Marginal PMF of X

$$p_X(x) = \sum_{y \in \text{Support}} p_{XY}(x,y)$$

$$= \begin{cases} \frac{1}{8} & \text{if } x=0 \\ \frac{3}{8} & \text{if } x=1 \\ \frac{3}{8} & \text{if } x=2 \\ \frac{1}{8} & \text{if } x=3 \end{cases}$$

Marginal CDF:

Let (X,Y) be a bivariate RV with CDF $F_{XY}(x,y)$.

... marginal CDF ...

The marginal CDF of

RV X is defined as

(i) if (X, Y) is discrete

$$F_X(x) = \lim_{y \to \infty} F_{XY}(x, y)$$

(ii) if (X, Y) is continuous

$$F_X(x) = \lim_{y \to \infty} F_{XY}(x, y) \, dy$$

Ex: Continuous bivariate RV

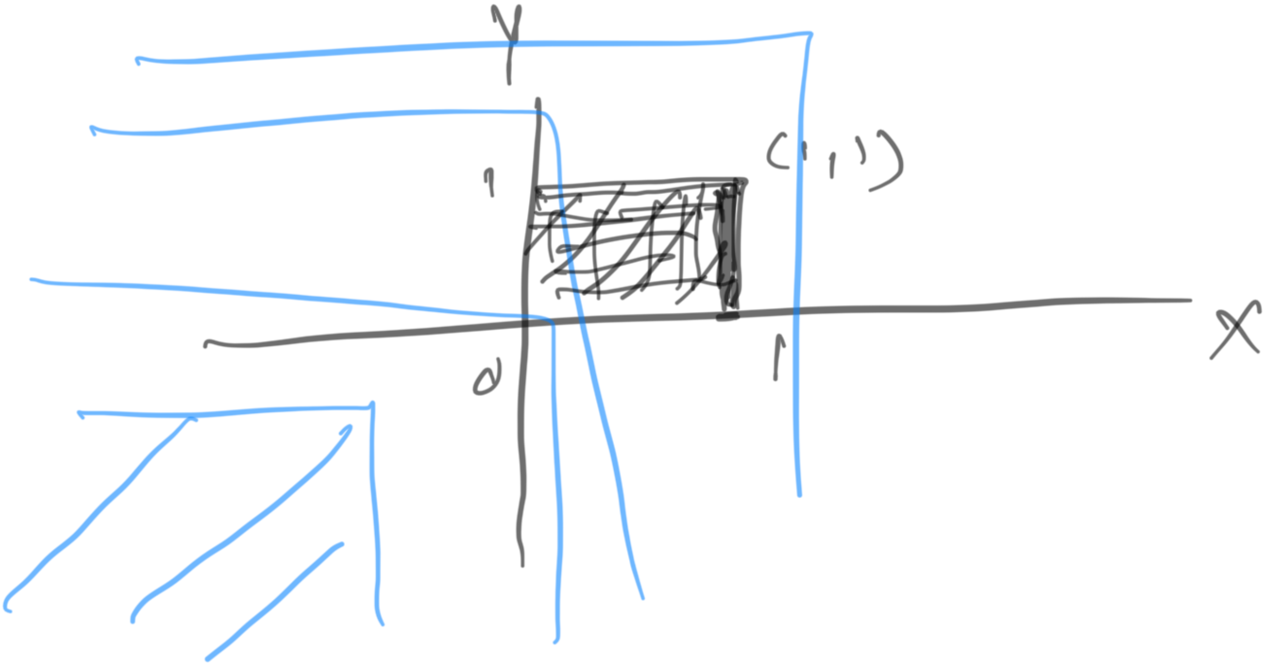
Let (X, Y) be a bivarite RV with CDF

$$F(x, y) = \int^{x} \int^{y} f(x, y) \, dx \, dy$$

$$F_{XY}(x,y) = \int_{-\infty} \int_{-\infty} f_{XY}$$

where

$$\underline{f(x,y)} = \begin{cases} 1 & \text{if } x \in (0,1), \ y \in (0,1) \\ 0 & \text{otherwise} \end{cases}$$



Joint PDF (Continuous RV)

let (X,Y) be a bivariate continuous RV with CDF

$$F(x,y) = \int^{x} \int^{y} f_{XY}(x,y)\, dy\, dx$$

f_{XY} ... $\int_{-\infty} \int_{-\infty}$

Then $f_{XY}(x,y)$ is said to be joint PDF of (X,Y).

Marginal PDF :

Marginal CDF of X is defined as

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\, dy$$

Eg: $f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\, dy$

$\int_0^1$ ... $x \in (0,1)$

$$= \int_0 1 \, dy, \; x \in (0,1)$$

$$= \begin{cases} 1, & x \in \underline{(0,1)} \\ 0, & \text{otherwise} \end{cases}$$

Multivariate RVs

Let a function $(X_1, \dots X_n)$:

$(\Omega, \mathcal{F}, P) \to (\mathbb{R}^n, \mathbb{B}(\mathbb{R}^n))$,

then $(X_1, \dots X_n)$ is said to be a multivariate RV.

CDF

$F_{X_1, \ldots X_n}(x_1, \ldots, x_n)$

$= \mathbb{P}(X_1 \le x_1, \ldots, X_n \le x_n)$

PMF & PDF (Joint)

Marginal PMF

Let $(X_1, \ldots X_n)$ be a discrete multivariate RV with joint CDF

$F_{X_1, \ldots X_n}(x_1, \ldots x_n)$ and

PMF

$p_{X_1 \ldots X_n}(x_1 \ldots x_n)$

$= \mathbb{P}(X_1 = x_1, \ldots, X_n = x_n)$

$= \mathbb{P}(X_1 = x_1, \ldots$

(I) Then marginal CDF of X_1 is defined as

$$F_{X_1}(x_1) = \lim_{\substack{x_2 \to \infty \\ x_3 \to \infty \\ \vdots \\ x_n \to \infty}} F_{X_1, \ldots X_n}(x_1, \ldots x_n)$$

(II) marginal CDF of X_1 and X_2 is defined as

$$F_{X_1 X_2}(x_1, x_2) = \lim_{\substack{x_3 \to \infty \\ x_4 \to \infty \\ \vdots \\ x_n \to \infty}} F_{X_1 \ldots X_n}(x_1, \ldots x_n)$$

(III) marginal PMF of X_1 is

$$p_{X_1}(x_1) = \sum \quad p_{X_1, \ldots X_n}(x_1, \ldots x_n)$$

$$p_{X_1}(x_1) = \sum_{x_2, x_3, \ldots x_n} \ldots X_1, \ldots X_n$$

(iv) marginal PMF of X_1 & X_2 is

$$p_{X_1, X_2}(x_1, x_2) = \sum_{x_3, x_4, \ldots} p_{X_1 \ldots X_n}(x_1, \ldots x_n)$$

Marginal PDF (cont).

Let $(X_1, \ldots X_n)$ be cont multivariate RV with CDF $F_{X_1 \ldots X_n}(x_1, \ldots x_n)$ and PDF $f_{X_1 \ldots X_n}(x_1, \ldots x_n)$.

Then

(1) ~~marginal CDF of X_i~~

~~is defined~~

~~↳~~ replace $\sum$ with $\int$

Independent RVs

~~Def~~ Let $(X_1, \dots, X_n)$ be multivariate RV with CDF

$F_{X_1 \dots X_n}(x_1, \dots x_n)$.

(1) The RVs $X_1, \dots X_n$ are independent if

$F_{X_1 \dots X_n}(x_1, \dots x_n)$

$= F_{X_1}(x_1) \cdots F_{X_n}(x_n)$

(ii) If $(X_1, \ldots X_n)$ is discrete then independence gives

$$p^{(PMF)}_{X_1 \ldots X_n}(x_1, \ldots x_n)$$

$$= p_{X_1}(x_1) \cdots p_{X_n}(x_n)$$

(iii) If $(X_1, \ldots X_n)$ is continuous then in terms of PDF independence reduces to

$$f_{X_1 \ldots X_n}(x_1, \ldots x_n)$$

$$= f_{X_1}(x_1) \cdots f_{X_n}(x_n).$$

$$= f_{X_1}(x_1) \cdots f_{X_n}(x_n)$$