# MTL108
# Common Continuous Random Variables

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Continuous random variables are defined over a continuous range and are described by probability density functions (PDFs).

## Uniform Distribution

**Definition 1** (Uniform Distribution)**.** A random variable $X$ following continuous uniform distribution over $(a, b)$, where $a < b$, is denoted by $X \sim \text{Uniform}(a, b)$ and its PDF is given by

$$f_X(x) = \begin{cases} \frac{1}{b-a}, & \text{if} \quad a \leq x \leq b, \\ 0, & \text{otherwise.} \end{cases}$$

Uniform distribution has a constant PDF over the support.

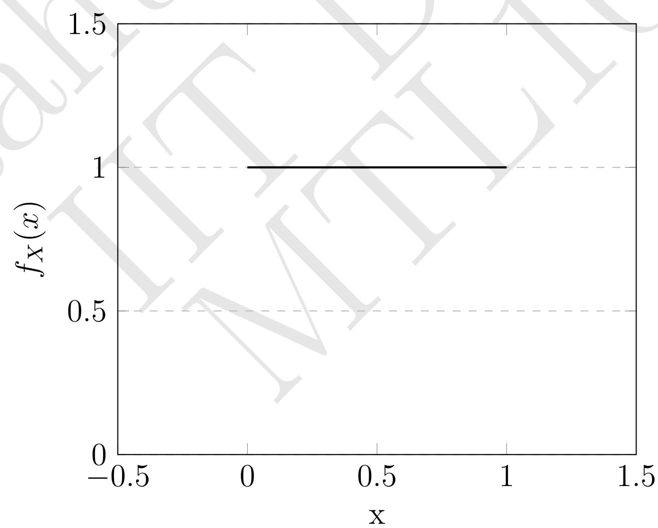


Figure 1: PDF for Uniform(0, 1).

- **Mean:**

$$\mathbb{E}[X] = \int_a^b x \cdot \frac{1}{b-a}\, dx = \frac{1}{b-a} \left[\frac{x^2}{2}\right]_a^b = \frac{b^2 - a^2}{2(b-a)} = \frac{b+a}{2}.$$

- **Variance:**

$$\mathbb{E}[X^2] = \int_a^b x^2 \cdot \frac{1}{b-a}\, dx = \frac{1}{b-a} \left[\frac{x^3}{3}\right]_a^b = \frac{b^3 - a^3}{3(b-a)} = \frac{(b-a)(b^2 + ab + a^2)}{3(b-a)} = \frac{b^2 + ab + a^2}{3},$$

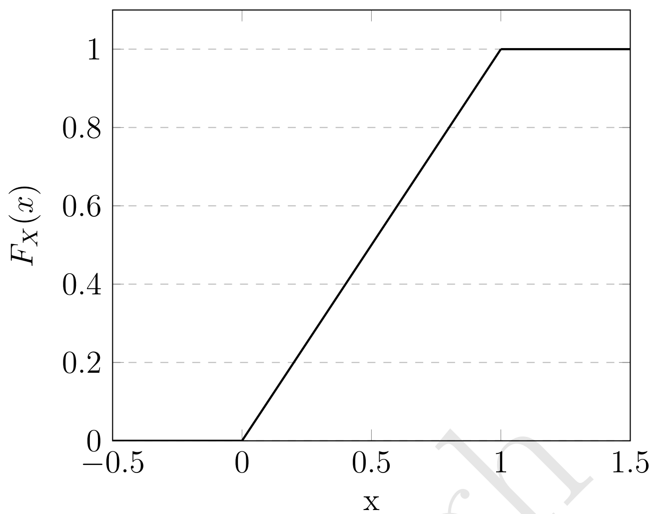


Figure 2: CDF for Uniform(0, 1).

Thus,

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \frac{b^2 + ab + a^2}{3} - \left(\frac{b+a}{2}\right)^2 = \frac{(b-a)^2}{12}.$$

**Example 1** (Application)**.** In quality control, if a machine produces parts with lengths uniformly distributed between 10 cm and 12 cm, the average length is $\frac{10+12}{2} = 11$ cm, and the variance is $\frac{(12-10)^2}{12} = \frac{1}{3} \approx 0.333$ cm$^2$, helping predict consistency.

## Exponential Distribution

**Definition 2** (Exponential Distribution)**.** A random variable $X$ is said to follow exponential distribution with rate $\lambda > 0$ is denoted by $X \sim \mathrm{Exponential}(\lambda)$ if its PDF has the form

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if} \quad x \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

- **Mean:**

$$\mathbb{E}[X] = \int_0^\infty x\lambda e^{-\lambda x}\, dx.$$

Use integration by parts: let $u = x$, $dv = \lambda e^{-\lambda x} dx$, so $du = dx$, $v = -e^{-\lambda x}$,

$$= [-xe^{-\lambda x}]_0^\infty + \int_0^\infty e^{-\lambda x}\, dx = 0 + \left[-\frac{e^{-\lambda x}}{\lambda}\right]_0^\infty = \frac{1}{\lambda}.$$

- **Variance:**

$$\mathbb{E}[X^2] = \int_0^\infty x^2 \lambda e^{-\lambda x}\, dx.$$

Integration by parts: $u = x^2$, $dv = \lambda e^{-\lambda x} dx$, so $du = 2x dx$, $v = -e^{-\lambda x}$,

$$\mathbb{E}[X^2] = [-x^2 e^{-\lambda x}]_0^\infty + 2\int_0^\infty x e^{-\lambda x}\, dx = 0 + 2 \cdot \frac{1}{\lambda} \cdot \frac{1}{\lambda} = \frac{2}{\lambda^2},$$

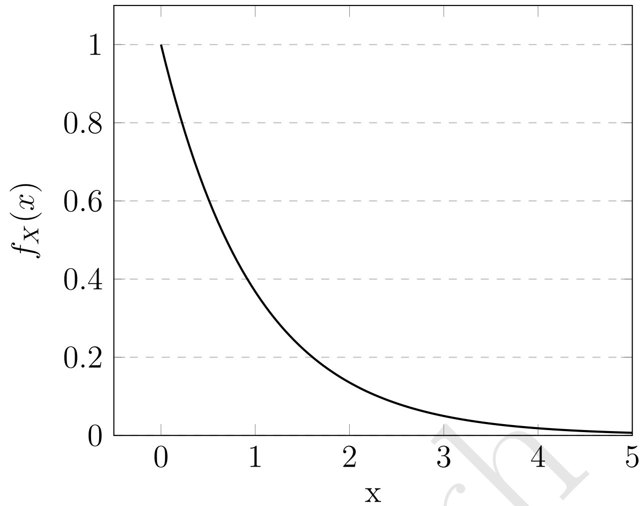


Figure 3: PDF for Exponential(1).

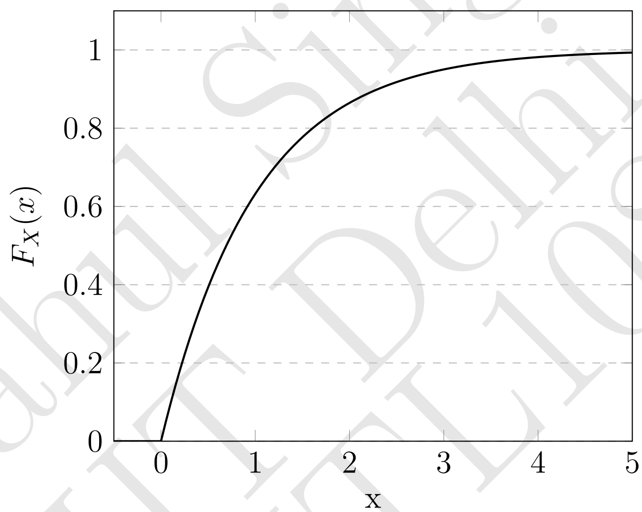


Figure 4: CDF for Exponential(1).



Thus,

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}.$$

**Example 2** (Application)**.** In reliability engineering, the time until a light bulb fails might follow an Exponential distribution with $\lambda = 0.1$ (failures per hour). The mean lifetime is $1/0.1 = 10$ hours, aiding in maintenance scheduling.

**Example 3** (Application)**.** The time until a radioactive particle decays follows an exponential distribution with $\lambda = 0.1$ per minute. The expected decay time is 10 minutes, with variance 100.

## Normal (Gaussian) Distribution

**Remark 1.** A fact from calculus: $\int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2}\, dz = \sqrt{2\pi}$.

**Definition 3** (Normal (Gaussian) Distribution)**.** A random variable $X$ is said to follow normal distribution with parameters $(\mu, \sigma^2)$ if its PDF has the form

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad -\infty < x < \infty.$$

It is denoted by $X \sim \mathcal{N}(\mu, \sigma^2)$.

- **Mean:**

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x \cdot \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, dx.$$

Substitute $z = (x-\mu)/\sigma$, so $x = \sigma z + \mu$, $dx = \sigma dz$,

$$\mathbb{E}[X] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\sigma z + \mu) e^{-z^2/2}\, dz = \mu \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2/2}\, dz + \sigma \cdot 0 = \mu,$$

since the odd function $ze^{-z^2/2}$ integrates to 0 over symmetric limits.

- **Variance:** Second moment

$$\mathbb{E}[X^2] = \int_{-\infty}^{\infty} x^2 \cdot \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, dx.$$

So, using transform $z = (x-\mu)/\sigma$, i.e., $x = \sigma z + \mu$, $dx = \sigma dz$, we have

$$\begin{aligned}
\mathbb{E}[X^2] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\sigma z + \mu)^2 e^{-z^2/2}\, dz \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\sigma^2 z^2 + 2\mu\sigma z + \mu^2) e^{-z^2/2}\, dz \\
&= \frac{1}{\sqrt{2\pi}} \left[ \int_{-\infty}^{\infty} \sigma^2 z^2 e^{-\frac{1}{2}z^2}\, dz + \int_{-\infty}^{\infty} 2\mu\sigma z e^{-\frac{1}{2}z^2}\, dz + \int_{-\infty}^{\infty} \mu^2 e^{-\frac{1}{2}z^2}\, dz \right].
\end{aligned}$$

Now, we evaluate each integral term.

- **Term 1:** $\int_{-\infty}^{\infty} \sigma^2 z^2 e^{-\frac{1}{2}z^2}\, dz = \sigma^2 \int_{-\infty}^{\infty} z^2 e^{-\frac{1}{2}z^2}\, dz$. Let

$$I = \int_{-\infty}^{\infty} z^2 e^{-\frac{1}{2}z^2}\, dz$$

We can rewrite this integral as:

$$I = \int_{-\infty}^{\infty} z \cdot (z e^{-\frac{1}{2}z^2})\, dz$$

Now, we apply the integration by parts formula, $\int u\, dv = uv - \int v\, du$. Let:

$$\begin{aligned}
u &= z \\
dv &= z e^{-\frac{1}{2}z^2}\, dz
\end{aligned}$$

Differentiating $u$ and integrating $dv$, we get:

$$du = 1\,dz = dz$$
$$v = \int ze^{-\frac{1}{2}z^2}\,dz$$

To solve the integral for $v$, we use a substitution. Let $w = -\frac{1}{2}z^2$, so $dw = -z\,dz$. This gives us:

$$v = \int e^{w}(-dw) = -e^{w} = -e^{-\frac{1}{2}z^2}$$

Substitute $u$, $v$, $du$, and $dv$ back into the integration by parts formula:

$$\int_{-\infty}^{\infty} u\,dv = [uv]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} v\,du$$
$$\int_{-\infty}^{\infty} z(ze^{-\frac{1}{2}z^2})\,dz = \left[z\left(-e^{-\frac{1}{2}z^2}\right)\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\left(-e^{-\frac{1}{2}z^2}\right)\,dz$$
$$= \left[-ze^{-\frac{1}{2}z^2}\right]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2}\,dz$$

Now we evaluate the first term at the limits of integration. We need to find the limit of $-ze^{-\frac{1}{2}z^2}$ as $z \to \pm\infty$. We can write this as $\lim_{z\to\pm\infty}\left(-\frac{z}{e^{\frac{1}{2}z^2}}\right)$. As $z \to \infty$, the exponential function in the denominator grows much faster than the linear function in the numerator, so the limit is 0. Similarly, as $z \to -\infty$, the limit is also 0.

$$\left[-ze^{-\frac{1}{2}z^2}\right]_{-\infty}^{\infty} = \left(\lim_{z\to\infty} -ze^{-\frac{1}{2}z^2}\right) - \left(\lim_{z\to-\infty} -ze^{-\frac{1}{2}z^2}\right) = 0 - 0 = 0$$

The integral simplifies to:

$$\int_{-\infty}^{\infty} z^2 e^{-\frac{1}{2}z^2}\,dz = \int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2}\,dz$$

We know that $\int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2}\,dz = \sqrt{2\pi}$. Substituting this into our result:

$$\int_{-\infty}^{\infty} z^2 e^{-\frac{1}{2}z^2}\,dz = \sqrt{2\pi}$$

- **Term 2:** $\int_{-\infty}^{\infty} 2\mu\sigma ze^{-\frac{1}{2}z^2}\,dz = 2\mu\sigma\int_{-\infty}^{\infty} ze^{-\frac{1}{2}z^2}\,dz = 0$, because the integrand $ze^{-\frac{1}{2}z^2}$ is an odd function, and the integration limits are symmetric around 0.
- **Term 3:** $\int_{-\infty}^{\infty} \mu^2 e^{-\frac{1}{2}z^2}\,dz = \mu^2\int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2}\,dz = \mu^2\sqrt{2\pi}$, because $\int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2}\,dz = \sqrt{2\pi}$.

Substitute the evaluated integrals back into the expression,

$$\mathbb{E}[X^2] = \frac{1}{\sqrt{2\pi}}\left[\sigma^2(\sqrt{2\pi}) + 2\mu\sigma(0) + \mu^2(\sqrt{2\pi})\right]$$
$$= \frac{1}{\sqrt{2\pi}}(\sigma^2\sqrt{2\pi} + \mu^2\sqrt{2\pi})$$
$$\Rightarrow \mathbb{E}[X^2] = \frac{\sqrt{2\pi}}{\sqrt{2\pi}}(\sigma^2 + \mu^2) = \sigma^2 + \mu^2.$$

Thus,

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \sigma^2 + \mu^2 - \mu^2 = \sigma^2.$$

**Remark 2.** If $X \sim \mathcal{N}(\mu, \sigma^2)$ then mean is the first parameter $\mu$ and variance is the second parameter $\sigma^2$. So, $X \sim \mathcal{N}(\mu, \sigma^2)$ amounts to saying that $X$ follows normal distribution with mean $\mu$ and variance $\sigma^2$.

**Definition 4** (Standard Normal Distribution)**.** If $Z \sim \mathcal{N}(0, 1)$ then we say that $Z$ is a standard normal random variable.

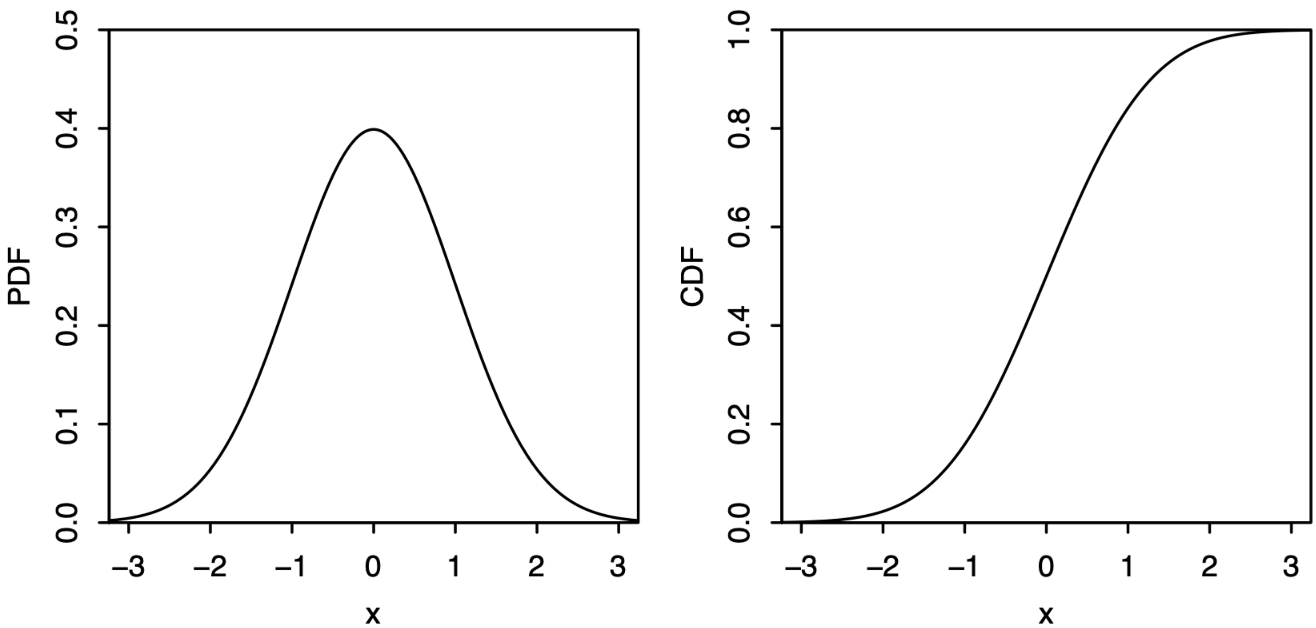


Figure 5: PDF and CDF of $Z \sim \mathcal{N}(0, 1)$

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**Theorem 1.** *If $X \sim \mathcal{N}(\mu, \sigma^2)$ then $Z = (X - \mu)/\sigma$ is a standard normal random variable.*

**Proof.** Observe that for any $a \in \mathbb{R}$,

$$\begin{aligned} \mathbb{P}(Z \leq a) &= \mathbb{P}\left(\frac{X - \mu}{\sigma} \leq a\right) \\ &= \mathbb{P}\left(X \leq \mu + a\sigma\right) \\ &= \int_{-\infty}^{\mu + a\sigma} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, dx. \end{aligned}$$

In the integral, substitute $x = \mu + z\sigma$, so $z = (x - \mu)/\sigma$, $dx = \sigma\, dz$ and $dx$ will be replaced with $\sigma\, dz$. Also, lower bound of integral at $x = \infty$, $z = -\infty$ and the upper bound at $x = \mu + a\sigma$, $z = a$. Thus,

$$\begin{aligned} \mathbb{P}(Z \leq a) &= \int_{-\infty}^{\mu + a\sigma} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, dx \\ \Rightarrow \mathbb{P}(Z \leq a) &= \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\, dz \quad \text{for any} \quad a \in \mathbb{R}. \end{aligned}$$

Therefore $Z \sim \mathcal{N}(0,1)$. $\square$

**Theorem 2.** *For a standard normal random variable $Z$,*
- **Mean:** $\mathbb{E}[Z] = 0$.
- **Variance:** $Var(Z) = 1$.

**Theorem 3.** *The PDF of the standard normal distribution is symmetric about $0$, i.e., if $Z \sim \mathcal{N}(0,1)$ then*

$$f_Z(-z) = f_Z(z), \quad \forall z \in \mathbb{R}.$$

**Proof.** Observe that

$$f_Z(-z) = \frac{1}{\sqrt{2\pi}} e^{-(-z)^2/2} = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} = f_Z(z).$$

Hence, the PDF is symmetric about 0. $\square$

**Theorem 4.** *For the standard normal distribution,*

$$\mathbb{P}(Z > z) = \mathbb{P}(Z < -z), \quad \forall z \geq 0.$$

**Proof.** Observe that

$$\mathbb{P}(Z > z) = \int_z^\infty f_Z(x)\, dx,$$

$$\mathbb{P}(Z < -z) = \int_{-\infty}^{-z} f_Z(x)\, dx.$$

Using the substitution $u = -x$ in the second integral:

$$\int_{-\infty}^{-z} f_Z(x)\, dx = \int_\infty^z f_Z(-u)(-du).$$

Since $f_Z(-u) = f_Z(u)$ by symmetry:

$$\int_{-\infty}^{-z} f_Z(x)\, dx = \int_z^\infty f_Z(u)\, du = \mathbb{P}(Z > z).$$

Thus,

$$\mathbb{P}(Z > z) = \mathbb{P}(Z < -z).$$

$\square$

**Remark 3.** The CDF of the standard normal distribution does not have a closed form and it is denoted by $\Phi$, that is,

$$\Phi(z) = \int_{-\infty}^z \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\, dz.$$

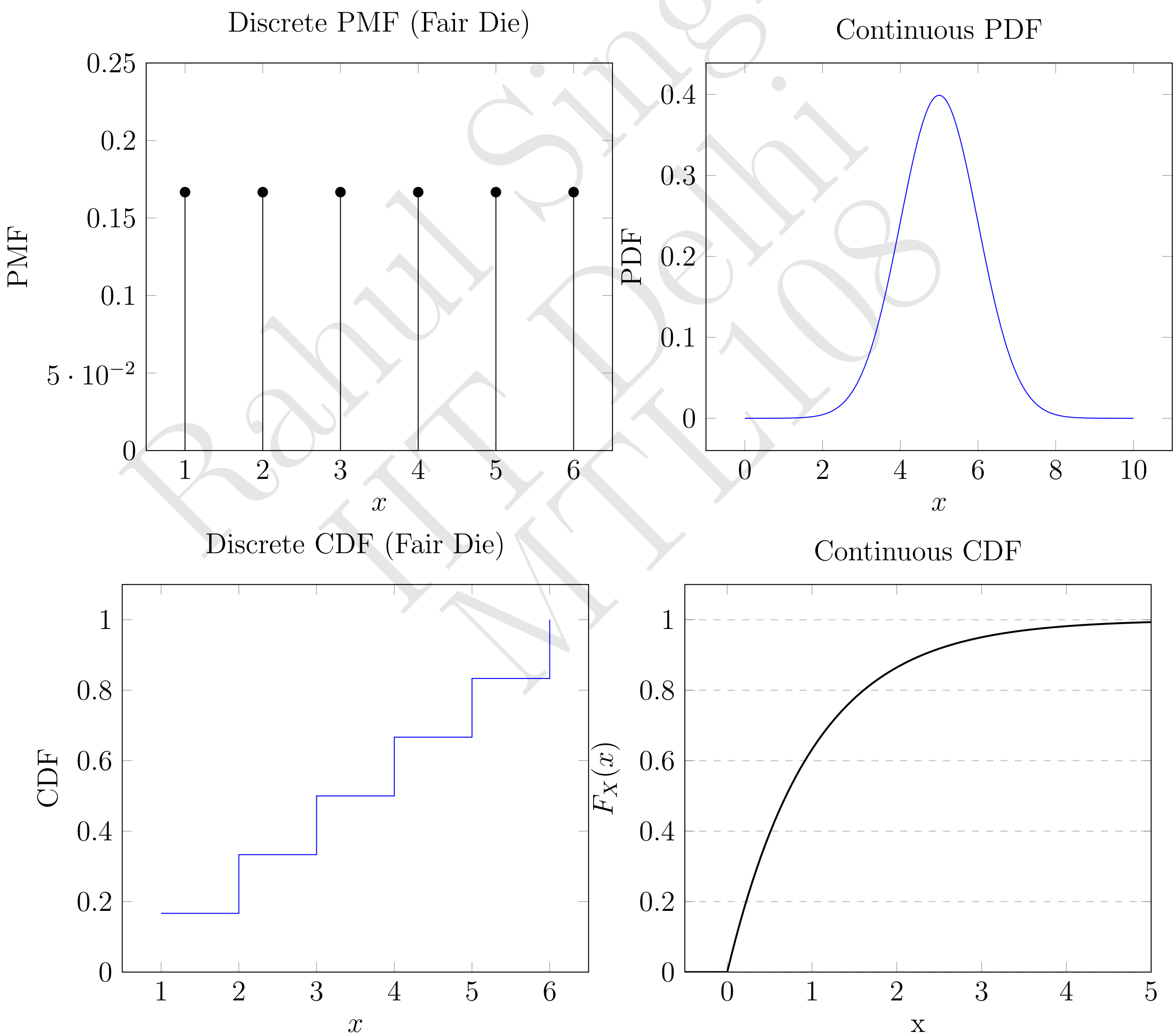
Therefore above theorem gives,

$$\Phi(-z) = 1 - \Phi(z), \quad \forall z \in \mathbb{R}.$$

**Example 4** (Application)**.** The normal distribution is a fundamental statistical concept applied widely in real-life scenarios, from the natural sciences to business. It's used because many phenomena, like human height, blood pressure, and standardized test scores, naturally cluster around an average value, with fewer data points appearing at the extremes.

For example, manufacturers rely on it for quality control, predicting the percentage of products that fall within acceptable specifications. In finance, it helps analyze stock market behavior and assess risk, though it has limitations for extreme market events. Even in education, it's used to compare test scores and standardize performance relative to peers. Thanks to the Central Limit Theorem, the average of many random variables tends toward a normal distribution, making it an invaluable tool for making predictions and understanding the world.

## Discrete vs Continuous RVs






- **PMF vs PDF:** The PMF (discrete) places positive probability mass at integer points; the

PDF (continuous) is a density — its values can exceed 1, and probabilities are given by areas under the curve, not point heights.

- **CDF differences:** The discrete CDF is a step function with jumps at integers equal to the PMF values; the continuous CDF is continuous and differentiable (its derivative is the PDF).
- **Probability at a point:** For a continuous distribution, $\Pr(X = x) = 0$ for any single point $x$. For a discrete distribution, $\Pr(X = k) = p_k > 0$ may hold.

| Aspect | Discrete | Continuous |
|---|---|---|
| Values | Countable (e.g., integers) | Uncountable (e.g., reals in interval) |
| Probability Function | PMF $p_X(x)$: $P(X = x)$ | PDF $f_X(x)$: density, not probability |
| Probability at Point | Can be positive | Always 0 |
| CDF | Step function | Continuous and smooth |
| Expectation | Sum: $\sum x p_X(x)$ | Integral: $\int x f_X(x)\, dx$ |
| Variance | Sum: $\sum (x-\mu)^2 p_X(x)$ | Integral: $\int (x-\mu)^2 f_X(x)\, dx$ |
| Examples | Coin flips, die rolls | Heights, waiting times |

## Discrete distributions summary

| Distribution | Notation | PMF $P(X = k)$ | Mean | Variance | MGF $M_X(t)$ |
|---|---|---|---|---|---|
| **Bernoulli** | $X \sim \text{Bernoulli}(p)$ | $\begin{cases} p, & k = 1 \\ 1-p, & k = 0 \end{cases}$ | $p$ | $p(1-p)$ | $(1-p) + pe^t$ |
| **Binomial** | $X \sim \text{Binomial}(n, p)$ | $\binom{n}{k} p^k (1-p)^{n-k},\ k = 0, \ldots, n$ | $np$ | $np(1-p)$ | $\big((1-p) + pe^t\big)^n$ |
| **Poisson** | $X \sim \text{Poisson}(\lambda)$ | $\frac{e^{-\lambda}\lambda^k}{k!},\ k = 0, 1, \ldots$ | $\lambda$ | $\lambda$ | $\exp\big(\lambda(e^t - 1)\big)$ |
| **Geometric** | $X \sim \text{Geometric}(p)$ | $(1-p)^{k-1} p,\ k = 1, 2, \ldots$ | $\frac{1}{p}$ | $\frac{1-p}{p^2}$ | $\frac{pe^t}{1-(1-p)e^t},\ t < -\ln(1-p)$ |
| **Negative Binomial** | $X \sim \text{NegBin}(r, p)$ | $\binom{k+r-1}{k} p^r (1-p)^k, \quad k = 0, 1, 2, \ldots$ | $\frac{r(1-p)}{p}$ | $\frac{r(1-p)}{p^2}$ | $\left(\frac{p}{1-(1-p)e^t}\right)^r,\ t < -\ln(1-p).$ |
| **Uniform Discrete** | $X \sim \text{UnifDis}(\{x_1, \ldots, x_m\})$ | $1/m,\ k = x_1, \ldots, x_m$ | $\frac{1}{m}\sum_{i=1}^m x_i$ | $\frac{1}{m}\sum_{i=1}^m x_i^2 - \big(\frac{1}{m}\sum_{i=1}^m x_i\big)^2$ | $\frac{1}{m}\sum_{i=1}^m e^{tx_i}$ |
| **Uniform Discrete** | $X \sim \text{UnifDis}(\{1, \ldots, m\})$ | $1/m,\ k = 1, \ldots, m$ | $(m+1)/2$ | $(m^2-1)/12$ | $\frac{1}{m}\sum_{i=1}^m e^{it}$ |

## Continuous distributions summary

| **Distribution** | Notation | PDF $f_X(x)$ | Mean | Variance | MGF $M_X(t)$ |
|---|---|---|---|---|---|
| **Uniform** | $X \sim \text{Uniform}(a, b)$ | $\begin{cases} \frac{1}{b-a}, & \text{if } \quad a \leq x \leq b, \\ 0, & \text{otherwise.} \end{cases}$ | $(b+a)/2$ | $(b-a)^2/12$ | $\frac{e^{tb}-e^{ta}}{t(b-a)}$, $t \neq 0$ |
| **Exponential** | $X \sim \text{Exponential}(\lambda)$ | $\begin{cases} \lambda e^{-\lambda x}, & \text{if } \quad x \geq 0, \\ 0, & \text{otherwise.} \end{cases}$ | $1/\lambda$ | $1/\lambda^2$ | $\frac{\lambda}{\lambda - t}$, $t < \lambda$ |
| **Normal** | $X \sim \mathcal{N}(\mu, \sigma^2)$ | $\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$, $-\infty < x < \infty$ | $\mu$ | $\sigma^2$ | $e^{\mu t + \frac{\sigma^2 t^2}{2}}$ |
| **Standard Normal** | $Z \sim \mathcal{N}(0, 1)$ | $\frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$, $-\infty < z < \infty$ | 0 | 1 | $e^{\frac{t^2}{2}}$ |

## Degenerate random variable

**Definition 5.** A random variable $X$ is said to be degenerate at $a \in \mathbb{R}$, iff

$$\mathbb{P}(X = a) = 1.$$

A non-random number can be viewed as a degenerate random variable.

**Lemma 1.** *Let $X$ be a degenerate random variable at $a \in \mathbb{R}$. Then,*

1. $\mathbb{E}(X) = a$,
2. $Var(X) = 0$.

## Symmetric random variable

**Definition 6.** A random variable $X$ is said to be symmetric about $a \in \mathbb{R}$, iff for all $x \in \mathbb{R}$,

$$\mathbb{P}(X \leq x - a) = \mathbb{P}(X \geq x - a).$$

If $a = 0$, the random variable is usually referred to as a symmetric random variable.

## Existence of expectation

**Definition 7.** Let $X$ be a random variable and $g : (\mathbb{R}, \mathbb{B}(\mathbb{R})) \to (\mathbb{R}, \mathbb{B}(\mathbb{R}))$ be a measurable function. The expectation $\mathbb{E}[g(X)]$ is said to exist if $\mathbb{E}[|g(X)|] < \infty$.

**Remark 4.** All functions used in practice are measurable, e.g., continuous, step function, etc.

### Examples: Discrete Random Variables with Support on Natural Numbers

**Random Variable $X_1$: Infinite Mean**

Let $X_1$ be a discrete random variable with support $\mathbb{N} = \{1, 2, 3, \ldots\}$ and probability mass function

$$p_1(x) = \frac{1}{c_1 x^2}, \quad \text{where } c_1 = \sum_{x=1}^{\infty} \frac{1}{x^2} = \zeta(2) = \frac{\pi^2}{6}.$$

The mean of $X_1$ is:

$$E[X_1] = \sum_{x=1}^{\infty} x \cdot p_1(x) = \frac{1}{c_1} \sum_{x=1}^{\infty} \frac{x}{x^2} = \frac{1}{c_1} \sum_{x=1}^{\infty} \frac{1}{x}.$$

By the $p$-test, the series $\sum_{x=1}^{\infty} \frac{1}{x^p}$ converges if $p > 1$ and diverges if $p \leq 1$. Here $p = 1$, so the harmonic series diverges to infinity. Hence $E[X_1] = \infty$.

**Random Variable $X_2$: Finite Mean but Infinite Second Moment**

Now define $X_2$ with support $\mathbb{N}$ and probability mass function

$$p_2(x) = \frac{1}{c_2 x^3}, \quad \text{where } c_2 = \sum_{x=1}^{\infty} \frac{1}{x^3} = \zeta(3).$$

The mean of $X_2$ is:

$$E[X_2] = \sum_{x=1}^{\infty} x \cdot p_2(x) = \frac{1}{c_2} \sum_{x=1}^{\infty} \frac{x}{x^3} = \frac{1}{c_2} \sum_{x=1}^{\infty} \frac{1}{x^2}.$$

Since $p = 2 > 1$, by the $p$-test the series converges to $\zeta(2) = \frac{\pi^2}{6}$. Thus $E[X_2] = \frac{\zeta(2)}{c_2}$ is finite.

However, consider $E[X_2^2]$:

$$E[X_2^2] = \sum_{x=1}^{\infty} x^2 \cdot p_2(x) = \frac{1}{c_2} \sum_{x=1}^{\infty} \frac{x^2}{x^3} = \frac{1}{c_2} \sum_{x=1}^{\infty} \frac{1}{x}.$$

Again, the harmonic series diverges ( $p = 1 \leq 1$ ), so $E[X_2^2] = \infty$. This implies that the variance of $X_2$ is also infinite.

# References


[1] Blitzstein, J. K., & Hwang, J. (2019). *Introduction to probability.* Chapman and Hall/CRC.


### Disclaimer