$(\Omega, \mathcal{F}, \mathbb{P})$

Ω — Sample space

$\mathcal{F}$ — Collection of events

$\mathbb{P}$ — Probability measure

$\mathbb{R}$ — real line

$\mathcal{B}(\mathbb{R})$ — Borel σ-field on $\mathbb{R}$

Problem: $\{1\} \in \mathcal{B}(\mathbb{R})$?

Theorem: (without proof)

$\mathcal{B}(\mathbb{R})$ is generated by

$\{(a, b] : -\infty < a \leq b < \infty\}$

(i) $\{(a, b) : \ldots$

(ii) $\{[a, b] : -\infty < a \leq b < \infty\}$

(iii) $\{[a, b) : -\infty < a \leq b < \infty\}$

(iv) $\{(a, b] : -\infty < a \leq b < \infty\}$

Hint:

$$\{1\} = \bigcap_{i=1}^{\infty} \left(1 - \frac{1}{i}, 1 + \frac{1}{i}\right)$$

Independent events:

Def: Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and

$A, B \in \mathcal{F}$ ... the events

$A, B \in \mathcal{F}$. The events A and B are said to be independent if

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$$

Def: Let $A_1, \ldots, A_n \in \mathcal{F}$.

The events $A_1, \ldots, A_n$ are said to be independent if for any subset $\{i_1, i_2, \ldots, i_k\} \subset \{1, 2, \ldots n\}$

$$\mathbb{P}(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k})$$

$$= \mathbb{P}(A_{i_1}) \cdot \mathbb{P}(A_{i_2}) \cdots \mathbb{P}(A_{i_k})$$

Def: Let $\{A_i\}_{i=1}^{\infty} \subseteq \mathcal{F}$.

Events $\{A_i\}^{\infty}$ are

events $\{A_i\}_{i=1}^{\infty}$ are said to be independent if every finite subset of $\{A_i\}_{i=1}^{\infty}$ is independent.

Remark:

# of events	independence
2	pairwise independence
>2	mutual independence

Eg: Two fair coins are tossed

$$\Omega = \{HH, HT, TH, TT\}$$

A_1 = head on first coin

A_2 = head on second coin

Are A_1 & A_2 independent?

Sol: $A_1 = \{HH, HT\}$

$A_2 = \{HH, TH\}$

$A_1 \cap A_2 = \{HH\}$

$\mathbb{P}(A_1 \cap A_2) = \frac{1}{4}$

$\mathbb{P}(A_1) = \frac{1}{2}$

$\mathbb{P}(A_2) = \frac{1}{2}$

$\mathbb{P}(A_1 \cap A_2) = \frac{1}{4}$

$= \frac{1}{2} \times \frac{1}{2}$

$= \mathbb{P}(A_1) \cdot \mathbb{P}(A_2)$

$\Rightarrow$ A_1 and A_2 are independent events.

Lemma: Mutual independence implies pairwise independence

Mean: If $A, B, C \in \mathcal{F}$ are independent then A, B are independent.

Lemma: Pairwise independence doesn't imply mutual independence.

Eg: $\Omega = \{1, 2, 3, 4\}$

$\mathcal{F} = 2^{\Omega}$

$\mathbb{P}\{\{\omega\}\} = \frac{1}{4}$ if $\omega \in \Omega$

$$\mathbb{P}(A) = \sum_{\omega \in \Omega \cap A} \mathbb{P}(\{\omega\})$$

$A_1 = \{1, 2\}$

$A_2 = \{2, 3\}$

$A_3 = \{1, 3\}$

$\rightarrow$ A_1, A_2 & A_3 are pairwise independent but they are not mutually independent.

Conditional probability:

Let $(\Omega, \mathcal{C}, \mathbb{P})$ be a probability space.

Def: Let $A \in \mathcal{C}$ such that $\mathbb{P}(A) > 0$. The conditional probability given A is defined as for any $B \in \mathcal{C}$

$$\underbrace{\mathbb{P}(B|A)}_{\text{probabilit}} = \frac{\mathbb{P}(B \cap A)}{\mathbb{P}(A)}$$

probability

B given A.

Theorem: Let $A \in \mathcal{F}$ such that $\mathbb{P}(A) > 0$. Then, $(\Omega, \mathcal{F}, \mathbb{P}(\cdot \mid A))$ is a probability space.

Proof:

Axiom 1: $\mathbb{P}(\cdot \mid A) \geq 0$

Axiom 2:

$$\mathbb{P}(\Omega \mid A) = \frac{\mathbb{P}(\Omega \cap A)}{\mathbb{P}(A)}$$

$$= 1$$

Axiom 3: Let $B_1, B_2, \ldots \in \mathcal{F}$

and $B_i \cap B_j = \phi$ for $i \neq j$.

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} B_i \,\middle|\, A\right)$$

$$= \frac{\mathbb{P}\left(\left(\bigcup_{i=1}^{\infty} B_i\right) \cap A\right)}{\mathbb{P}(A)}$$

$$= \frac{\mathbb{P}\left(\bigcup_{i=1}^{\infty} (B_i \cap A)\right)}{\mathbb{P}(A)}$$

Clearly $(B_i \cap A) \cap (B_j \cap A) = \phi$ for all $i \neq j$, so using 3rd axiom

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} (B_i \cap A)\right)$$

$$= \sum_{i=1}^{\infty} \mathbb{P}(B_i \cap A)$$

$i=1$

Therefore

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} B_i \,/\, A\right)$$

$$= \sum_{i=1}^{\infty} \frac{\mathbb{P}(B_i \cap A)}{\mathbb{P}(A)}$$

$$= \sum_{i=1}^{\infty} \mathbb{P}(B_i | A)$$

Lemma: Let $A, B \in \mathcal{E}$ such that $\mathbb{P}(A) > 0$. Further let A and B are independent then

$$\mathbb{P}(B|A) = \mathbb{P}(B)$$

$\mathbb{P}(B|A) = \mathbb{P}(B)$

<u>Proof</u> : easy