Jan 28

Let X be a RV with "finite second moment".

Then

Lemma: For any fixed $a, b \in \mathbb{R}$,

(i) $Var(aX) = a^2 Var(X)$

(ii) $Var(X+b) = Var(X)$

(iii) $Var(aX+b) = a^2 Var(X)$

Proof (i) $Var(aX)$

$$= \mathbb{E}\left[(aX)^2\right] - \left[\mathbb{E}(aX)\right]^2,$$

$$= \mathbb{E}[a^2 x^2] - [\mathbb{E}(ax)]^2$$

linearity of expectation

$$= a^2\,\mathbb{E}(x^2) - [a\,\mathbb{E}(x)]^2$$

$$= a^2\,\mathbb{E}(x^2) - a^2\,[\mathbb{E}(x)]^2$$

$$= a^2\left[\underbrace{\mathbb{E}(x^2) - [\mathbb{E}(x)]^2}\right]$$

$$= a^2 \; Var(x)$$

(ii) $Var(x+a)$

$$= \mathbb{E}\left[\underbrace{\left((x+a) - \mathbb{E}(x+a)\right)}^2\right]$$

$$= \mathbb{E}\left[\left(x+a - \mathbb{E}(x) - \mathbb{E}(a)\right)^2\right]$$

$$= E\left[(X + a - E(X) - a)^2\right]$$
$$= E\left[(X - E(X))^2\right]$$
$$= \mathrm{var}(X)$$

Standard Deviation

Def: let X be a RV with variance var(X). it's standard deviation is defined as
$$SD(X) = \sqrt{\mathrm{var}(X)}$$

Lemma: let X be RV.

with $var(X) = \sigma^2$ and $E(X) = \mu$. Then

(I) $E(X-\mu) = 0$

(II) $E\left(\frac{X-\mu}{\sigma}\right) = 0$

(III) $Var\left(\frac{X-\mu}{\sigma}\right) = 1$

$$X \longrightarrow \frac{X - E(X)}{\sqrt{var(X)}}$$

$$Y \longrightarrow \frac{Y - E(Y)}{\sqrt{var(Y)}}$$

Standardization ?

Negative Binomial dist

Negative Binomial dist

k heads

$(r+k)^{th}$ toss is head

among $(r+k-1)$ tosses

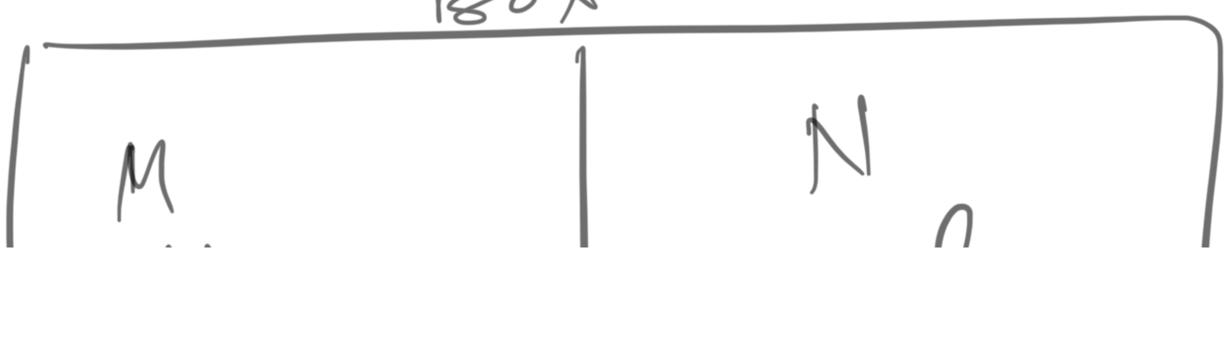
$\swarrow$ r tails $\searrow$ $(k-1)$ heads

$$\binom{r+k-1}{k-1} p^{k-1} (1-p)^{r} \cdot p$$

$$= \binom{r+k-1}{k-1} p^{k} (1-p)^{r}$$

Hyper geometric distribution

Box:



total $= M + N$

n balls are drawn randomly without replacement.

$X = \#$ of blue balls drawn

$\mathbb{P}(X = k)$

$= \mathbb{P}($ out of n balls drawn k are blue $)$

$$\binom{M}{k} \binom{N}{n-k}$$

$$= \frac{\binom{\cdots}{k}\binom{\cdots}{\cdots}}{\binom{M+N}{n}}$$

German tank problem

K = max. label of the tank captured.

River

Dolphin M	Other N.

n –

R -

Discrete uniform distribution

$S = \{a_1, a_2, \ldots, a_n\}$ ← finite

$\subseteq \mathbb{R}$

the PMF of 'discrete uniform RV X' with support S is defined as

$$\mathbb{P}(X = x) = \begin{cases} \frac{1}{n} & \text{if } x \in S \\ 0 & \text{otherwise} \end{cases}$$

① Mean

$$\mathbb{E}(X) = \sum_{x \in S} x \cdot \mathbb{P}(X = x)$$

$$= \frac{a_1 + a_2 + \dots + a_n}{n}$$

Eg:- $S = \{1, 2, 3, \dots, n\}$

Let X is discrete uniform RV on S.

Then

$$\mathbb{E}(X) = \frac{1 + 2 + \dots + n}{n}$$

$$= \frac{n+1}{2}$$

$$\mathbb{E}(X^2) = \sum x^2 \cdot \frac{1}{n}$$

$$\mathbb{E}(\;\;) \sum_{x \in S}$$

$$= \frac{1}{n} \sum_{x=1}^{n} x^2$$

$$= \frac{1}{n} \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{(n+1)(2n+1)}{6}$$

$$\mathrm{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2$$

$$= \frac{(n+1)(2n+1)}{6} - \left(\frac{n+1}{2}\right)^2$$

Common Continuous RVs

(1) Uniform RV:

X is said to be uniform RV on $(a,b) \subseteq \mathbb{R}$, if its PDF is given by

$$f(x) = \begin{cases} \frac{1}{b-a}, & x \in (a,b) \\ 0, & \text{otherwise} \end{cases}$$

clearly $f(x) \geq 0$

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

b

$$E(X) = \int_a^b x\, f(x)\, dx$$

$$= \frac{1}{b-a} \cdot \int_a^b x\, dx$$

$$= \frac{1}{b-a} \frac{b^2 - a^2}{2}$$

$$= \frac{a+b}{2}$$

$$E(X^2) =$$

$$var(X) =$$