March 17

If $X_1, \ldots, X_n$ are IID RVs from a distribution with mean μ and variance σ^2.

Then, the WLLN gives for

$$\left[\bar{X}_n = \frac{X_1 + \cdots + X_n}{n}\right]$$

$$\bar{X}_n \xrightarrow{p} \mu$$

CMT:

$$g(x) = x^2$$

$$g(\bar{X}_n) \longrightarrow g(\mu)$$

$$\Rightarrow \boxed{\bar{X}_n^2 \xrightarrow{p} \mu^2}$$

$$\boxed{\bar{X}_n^3} \xrightarrow{p} \mu^3$$

<u>CMT</u>

① If $X_n \xrightarrow{p/d} X$

and $g: \mathbb{R} \to \mathbb{R}$

a <u>continuous function</u>.

then $g(X_n) \xrightarrow{p/d} g(X)$

$g(X_n) \to \to g$

Think about Coefficient of Skewness

$$\frac{E((X-\mu)^3)}{\sigma^{3/2}} ? \checkmark$$

---

we have

$$S_n^2 = \underbrace{\frac{1}{n}\sum_{i=1}^{n} X_i^2}_{\xrightarrow{p} E(X_1^2)} - \bar{X}_n^2$$

$$\bar{X}_n^2 \xrightarrow{p} \mu^2$$

need → $\boxed{\text{Existence of fourth moment}}$

$$\mathrm{var}(X_1) = E(X_1^2) - [E(X_1)]^2$$

$$\Rightarrow \sigma^2 = E(X_1^2) - \mu^2$$

$$\Rightarrow E(X_1^2) = \sigma^2 + \mu^2$$

Substitute

$$S_n^2 = \underbrace{\frac{1}{n}\sum_{i=1}^{n} X_i^2}_{\xrightarrow{p}\ \sigma^2+\mu^2} - \underbrace{\bar{X}_n^2}_{\xrightarrow{p}\ \mu^2}$$

Using joint convergence in probability theorem

$$S_n^2 \xrightarrow{p} (\sigma^2+\mu^2) - \mu^2$$

$$\sigma^2$$

$= \sigma$

$$\Rightarrow S_n^2 \xrightarrow{p} \sigma^2$$

Function

$g(x, y) = x - y$

is a continuous function.

What about S_{n-1}^2?

$$S_{n-1}^2 = \frac{n}{n-1} S_n^2$$

$$= \frac{n}{n-1} \frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X})^2$$

$$\Rightarrow S_{n-1}^2 = \frac{n}{n-1} S_n^2$$

$\downarrow p \quad \downarrow p$

$1 \qquad \sigma^2$

$g(x) = cx$ for $c \in \mathbb{R}$

is contin

CMT

$$S_{n-1}^2 \xrightarrow{p} \sigma^2$$

Slutsky's Theorem (without proof)

Theorem: Let $X_n \xrightarrow{d} X$

and $Y_n \xrightarrow{p} c \neq 0$

Then, for continuous $g(\cdot)$,

$$g(X_n, Y_n) \xrightarrow{d} g(X, c)$$

Therefore

(I) $X_n \pm Y_n \xrightarrow{d} X \pm c$

(II) $X_n Y_n \xrightarrow{d} cX$

(III) $\dfrac{X_n}{Y_n} \xrightarrow{d} \dfrac{X}{c}$

CLT let $X_1, \ldots, X_n$ IID RVs with mean μ and variance σ^2.

~~Then~~, Define $\overline{X}_n = \dfrac{X_1 + \ldots + X_n}{n}$

Then,

$$\overline{X}_n - \mu \xrightarrow{d} N(0,1)$$

$$\frac{}{\sigma/\sqrt{n}} \longrightarrow N(\ldots$$

$$\Rightarrow \frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} \xrightarrow{d} N(0,1)$$

we know

$$S^2 \xrightarrow{p} \sigma^2$$

CMT

$$S := \sqrt{S^2} \xrightarrow{p} \sqrt{\sigma^2} = \sigma$$

$$\frac{\sigma}{S} \xrightarrow{p} 1$$

Slutsky's Theorem applicab

$$\frac{\sigma}{S} \cdot \frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} \xrightarrow{d} N(0,1)$$

$$\Rightarrow \frac{\sqrt{n}(\bar{X}_n - \mu)}{} \xrightarrow{d} N(0,1)$$

S

Multivariate normal distribution

Def: Let $\underset{\sim}{X} = (X_1, \dots, X_d)^T$.

$\underset{\sim}{X}$ is said to have multivariate normal distribution iff for all $a_1, \dots a_d \in \mathbb{R}$

$a_1 X_1 + \dots + a_d X_d$ has a normal distribution