March 27

$X_1, \ldots, X_n$ IID

Poisson(λ)

we saw, the MLE is

$$\hat{\lambda} = \bar{X} = \frac{X_1 + \cdots + X_n}{n}$$

For $X \sim$ Poisson(λ)

we know that

$$\mathrm{var}(X) = \lambda$$

Define

$$S_{n-1}^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2$$

We proved

$$\mathbb{E}(S_{n-1}^2) = Var(X)$$
$$= \lambda$$

$\Rightarrow$ S_{n-1}^2 is unbiased estimator of λ

but S_{n-1}^2 is the MLE

Discrete uniform

$X_1, X_2, \dots, X_n$ are IID from Discrete Uniform distribution with support

$\Omega = \{1, 2, \ldots, M\}$, $\cancel{M \in \mathbb{N}}$

So, the PMF is

$$\mathbb{P}(X = x) = \begin{cases} \frac{1}{M} & \text{if } x \in \Omega \\ 0 & \text{otherwise} \end{cases}$$

$\Omega = \{x \in \mathbb{N} : x \le m\}$

$$= \frac{1}{M} \mathbb{1}_{\Omega}(x)$$

Where

$$\mathbb{1}_{\Omega}(x) = \begin{cases} 1 & \text{if } x \in \Omega \\ 0 & \text{otherwise} \end{cases}$$

The likelihood function is

$$L(M) = \mathbb{P}(X_1 = x_1, \ldots, X_n = x_n)$$

$$= \prod_{i=1}^{n} \mathbb{P}(X_i = x_i)$$

$$= \prod_{i=1}^{n} \frac{1}{M} \mathbb{1}_{\Omega}(x_i)$$

$$\Rightarrow L(M) = \frac{1}{M^n} \prod_{i=1}^{n} \mathbb{1}_{\Omega}(x_i)$$

we want to find MLE of M

You have $\{x_1, \dots, x_n\}$

<u>Mode</u>

$\{1, 2, 3, 3, 5\}$

Mode $= 3 =: \tilde{M}_1$, some estimator

$L(\tilde{M}_1) = 0$

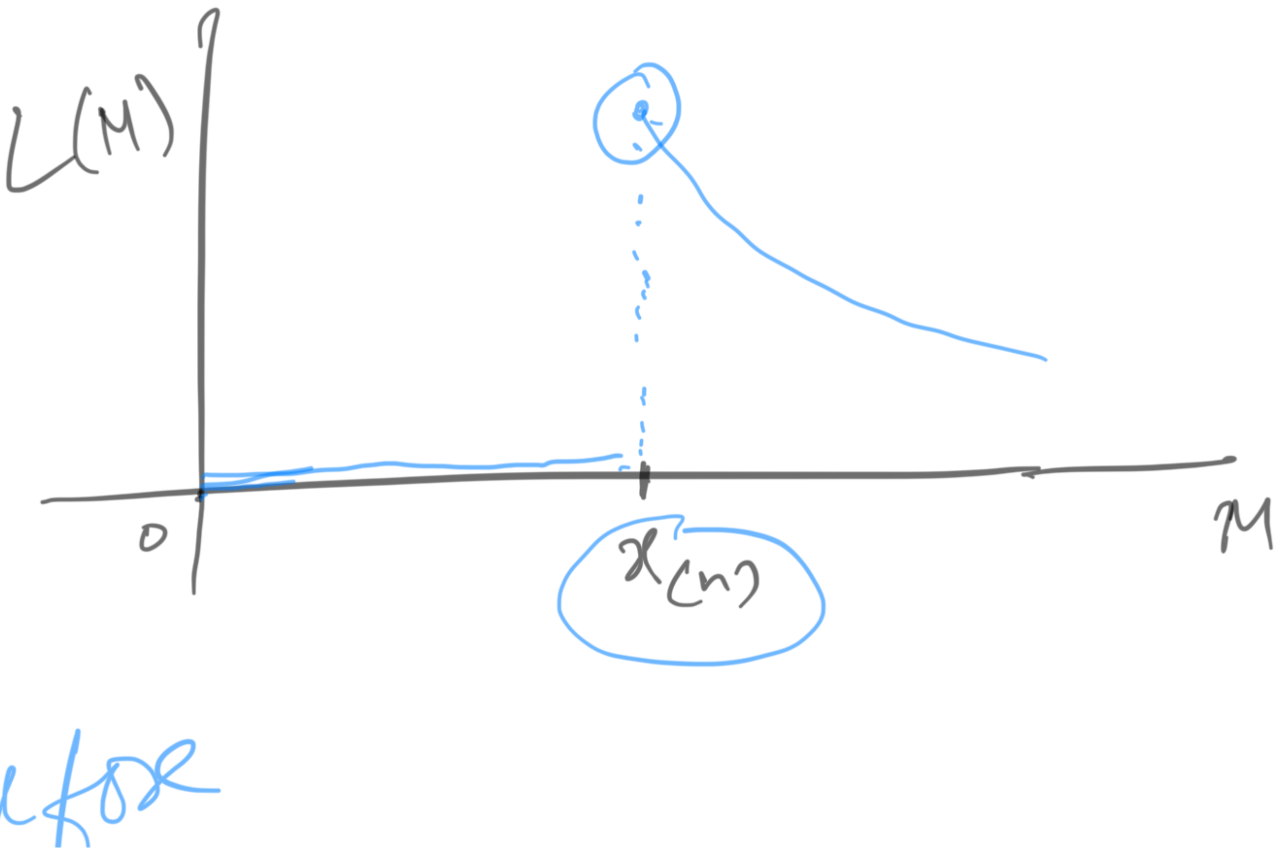
$\tilde{M}_2 = 6$,

$$L(\tilde{M}_2) = \frac{1}{\tilde{M}_2^5} = \frac{1}{6^5}.$$

$$L(\tilde{M}) = \frac{1}{\tilde{M}^5} \text{ if } \tilde{M} > \max\{x_1, \dots, x_n\}$$

Define

$$x_{(n)} = \max\{x_1, \dots, x_n\}$$



then for

(i) if $M < x_{(n)}$

then $L(M) = 0$

(ii) if $M \geq x_{(n)}$

then $L(M) = \frac{1}{M^n}$,

This is a decreasing function of M.

Therefore the $L(M)$ is maximised if

$M = x_{(n)}$

$= \max\{x_1, \ldots x_n\}$

So, the MLE is

$$\hat{M} = X_{(n)}$$

$$= \max\{X_1, \ldots, X_n\}$$

Next

let $X_1, \ldots, X_n$ be IID $U(0, \theta)$ oberrations (Continuous uniform). The PDF

$$\boxed{f(x) = \begin{cases} \frac{1}{\theta} & \text{if } x \leq \theta \\ 0 & \text{if } x > \theta \end{cases}}$$

The parameter space is

$\Theta = \{\theta : \theta > 0\}$

The likelihood function

$$L(\theta) = \prod_{i=1}^{n} f(x_i)$$

$$= \prod_{i=1}^{n} \frac{1}{\theta} \cdot \mathbb{1}(x_i \le \theta)$$

where $\mathbb{1}(x_i \le \theta) = \begin{cases} 1 & \text{if } x_i \le \theta \\ 0 & \text{otherwise} \end{cases}$

~~De~~fine $X_{(n)} = \max\{X_1, \dots, X_n\}$

Then

$$L(\theta) = \frac{1}{\theta^n} \cdot \mathbb{1}(X_{(n)} \le \theta)$$

because

$\{X_{(n)} \le \theta\}$

$= \{X_1 \le \theta, X_2 \le \theta, \ldots, X_n \le \theta\}$

therefore

$$L(\theta) = \frac{1}{\theta^n} \cdot \mathbb{1}(X_{(n)} \le \theta)$$

$$= \begin{cases} 0 & \text{if } \theta < X_{(n)} \\ \frac{1}{\theta^n} & \text{if } \theta \ge X_{(n)} \end{cases}$$

So, $L(\theta)$ is maximised

if $\theta = X_{(n)}$

$= \max\{X_1, \ldots X_n\}$

Thus the MLE is

$$\boxed{\hat{\theta} = \max\{x_1, \dots, x_n\}}$$

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let $X_1, \dots, X_n$ are IID obs. from a pop. with PDF

$$f(x) = \begin{cases} \frac{1}{\theta} & \text{if } 0 < x < \theta \\ 0 & \text{otherwise} \end{cases}$$

The likelihood function is

$$L(\theta) = \frac{1}{\theta^n} \cdot \mathbb{1}(X_{(n)} < \theta)$$

$$= \begin{cases} 0 & \text{if } \theta \leq X_{(n)} \\ \frac{1}{\theta^n} & \text{if } \theta > X_{(n)} \end{cases}$$

For some $\varepsilon > 0$,

if $\tilde{\theta} = X_{(n)} + \varepsilon$

Then

$$L(\tilde{\theta}) = \frac{1}{\tilde{\theta}^n}$$

ε cannot be determined based on given sample

So, the MLE does not exist.