Feb 17

Some variations of WLLN

Theorem: Let $X_1, X_2, \ldots$ be independent RVs with common mean μ and variance

$$\mathrm{Var}(X_i) = 1 + \frac{1}{i} \quad \text{for } i \in \mathbb{N}.$$

Then, $\bar{X}_n = \frac{1}{n}(X_1 + \cdots + X_n) \xrightarrow{P} \mu$.

Proof: Using linearity of expectation

$$\mathbb{E}(\bar{X}_n) = \mu$$

$$\mathrm{Var}(\bar{X}_n) = \mathrm{Var}\left(\frac{1}{n}(X_1 + X_2 + \cdots + X_n)\right)$$

$$= \frac{1}{n^2} \operatorname{Var}(X_1 + \cdots + X_n)$$

$$= \frac{1}{n^2} \sum_{i=1}^{n} \operatorname{Var}(X_i)$$

Observe that

$$\operatorname{var}(X_i) = 1 + \frac{1}{i}, \quad i \in \mathbb{N}$$

$$\leq 2$$

$$\operatorname{var}(\bar{X}_n) = \frac{1}{n^2} \sum_{i=1}^{n} \operatorname{var}(X_i)$$

$$\leq \frac{1}{n^2} \sum_{i=1}^{n} 2$$

$$= \frac{1}{n^2} \times 2n$$

$$\Rightarrow \boxed{\operatorname{var}(\bar{X}_n) \leq \frac{2}{n}}$$

Using Chebyshev's ... $\varepsilon > 0$.

Using Chebyshev's ,

$$\mathbb{P}\left(|\bar{X}_n - \mu| \geq \varepsilon\right)$$

$$\leq \frac{\operatorname{var}(\bar{X}_n)}{\varepsilon^2}$$

$$\leq \frac{2}{n\varepsilon^2} \longrightarrow 0$$

as $n \to \infty$

So, by def.

$$\bar{X}_n \xrightarrow[n\to\infty]{p} \mu$$

Theorem: Let $X_1, X_2, \ldots$ are identically distributed random variables with random variables

mean μ and variance σ^2.

Further let

$$\mathrm{Cov}(X_i, X_j) = c^{-|i-j|}$$

where $c > 1$.

Then,

$$\bar{X}_n = \frac{X_1 + \cdots + X_n}{n} \xrightarrow{p} \mu.$$

$$\boxed{\text{Use: } |\mathrm{Cov}(X,Y)|^2 \leq \mathrm{Var}(X) \cdot \mathrm{Var}(Y)}$$

Hint:

$$\mathrm{Var}(\bar{X}_n) = \frac{1}{n^2} \mathrm{Var}(X_1 + \cdots + X_n)$$

$$= \sum^{n} \ldots$$

$$= \frac{1}{n^2}\left[\sum_{i=1} Var(X_i) + 2\sum_{i<j} cov(X_i, X_j)\right]$$

Properties of Standard normal RV $N(0,1)$

PDF
$$\boxed{f(x) = \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}, \quad x \in \mathbb{R}}$$

... symmetric about 0.

$f(x)$ is symm

$$f(x) = f(-x)$$

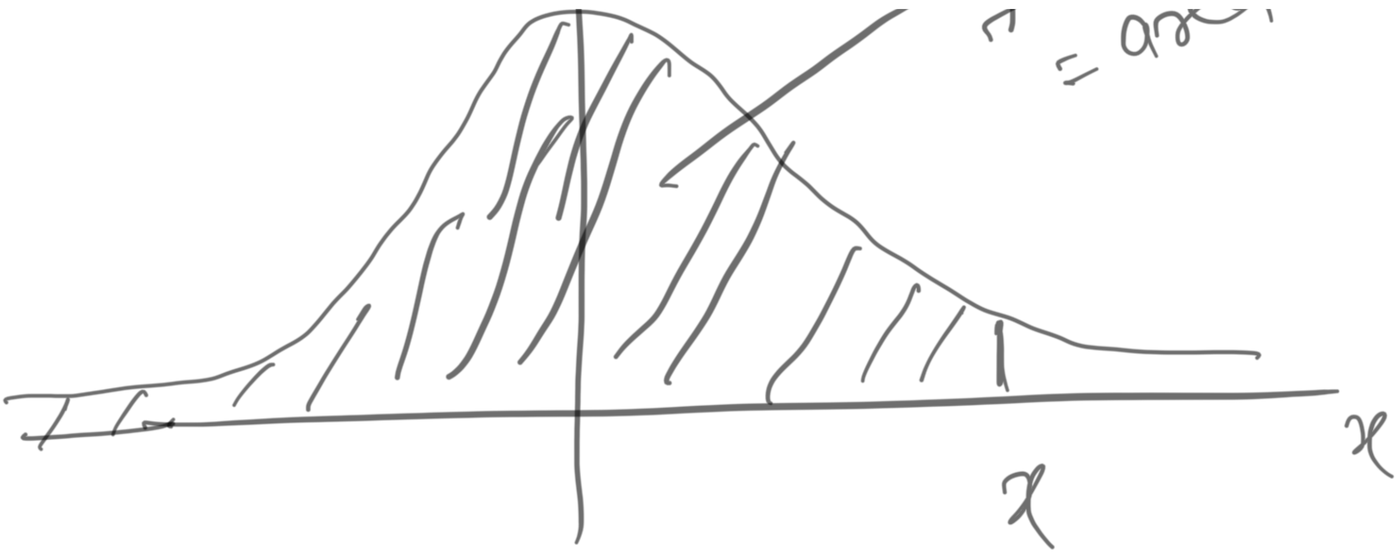
Notation: Usually PDF of $N(0,1)$ is denoted by ϕ.

CDF: CDF of $N(0,1)$ is denoted by $\Phi(\cdot)$.

$$\Phi(x) = \int_{-\infty}^{x} \phi(x)\, dx$$

$$= \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx$$

let $x > 0$



Lemma:

$$\boxed{\Phi(x) = 1 - \Phi(-x)}$$

Theorem: Let $X_1, X_2, \ldots$ be IID RVs with mean μ and variance σ^2 and finite and variance σ^2 and finite fourth moment.

Define $\bar{X}_n = \dfrac{X_1 + \cdots + X_n}{n}$

$$S_n^2 = \underbrace{\frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X}_n)^2}_{\text{sample variance}}$$

Then, $S_n^2 \xrightarrow{p} \sigma^2$.

<u>Hint:</u>

$$S_n^2 = \underbrace{\frac{1}{n} \sum_{i=1}^{n} X_i^2}_{} - (\bar{X}_n)^2$$

Continuous mapping <u>Theorem</u>:

if $T_n \xrightarrow{} \mu$ and g
is continuous function.
then, $g(T_n) \xrightarrow{p} g(\mu)$

Theorem: let $X_n \xrightarrow{p} \mu_1$
and $Y_n \xrightarrow{p} \mu_2$.
then $X_n + Y_n \xrightarrow{p} \mu_1 + \mu_2$

$$S_n^2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2 - (\bar{X}_n)^2$$

$$\frac{1}{n}\sum_{i=1}^{n} X_i^2 \xrightarrow{p} \mathbb{E}(X_1^2) \qquad (\bar{X}_n)^2 \xrightarrow{p} \mu^2$$

$$= \sigma^2 + \mu^2$$